# AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

A. B. COBLE
UNIVERSITY OF ILLINOIS

ABRAHAM COHEN
THE JOHNS HOPKINS UNIVERSITY

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RICE INSTITUTE

F. D. MURNAGHAN
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PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

VOLUME LIV
1932

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

Mathematics

JAN 11 1932

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THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $7.50 (foreign postage 25 cents); single numbers $2.00.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at The Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

# ON THE CONSTRUCTION OF CYCLIC ALGEBRAS WITH A GIVEN EXPONENT.

By A. ADRIAN ALBERT.

1. *Introduction.* The most important linear associative algebras at present are the cyclic (Dickson) algebras. These algebras are normal simple algebras of order $n^2$ (degree $n$) over a non-modular field $F$. They are defined by a cyclic field $Z$ of order $n$ over $F$ and a scalar $\gamma$ in $F$. The exponent of an algebra $A$ is the least integer $\rho$ for which the direct power $A^\rho$ is a total matric algebra, and it is desirable to know the conditions on $\gamma$ that $A$ have any given exponent. For algebras over an algebraic field $\Omega$ H. Hasse * has shown that the least integer $\sigma$ for which $\gamma^\sigma$ is the norm of a quantity of $Z$ is the exponent of the cyclic algebra $A$. This result was also used to show that $A$ is a division algebra if and only if

$$\gamma, \gamma^2, \cdots, \gamma^{n-1}$$

are all not the norms of any quantities of $Z$, a rather complicated set of conditions. For $n = 4$ the author recently simplified these conditions.†

The author has reduced essentially all problems on cyclic algebras to the case where $n = p^t$, $p$ a prime.‡ He shows here that the exponent of a cyclic algebra $A$ of degree $p^t$ is an integer $\tau = p^e$ such that $\gamma^\tau$ is the norm of a quantity of $Z$ while if $\pi = p^{e-1}$ then $\gamma^\pi$ is not the norm of any quantity of $Z$. This reduces the number of powers to be considered from $n - 1$ to *two*. But a far greater simplification is obtained.

A cyclic field $Z$ of order $p^t$ over $F$ contains a sequence of cyclic sub-fields

$$Z_t = Z, Z_{t-1}, \cdots, Z_1, Z_0 = F$$

where $Z_i$ has order $p^i$ over $F$. For each $Z_i$ we may evidently define a norm symbol $N_i(\ )$. The author has now been able to secure the remarkable result that

$$\gamma^{p^e} = N(f), \qquad f \text{ in } Z,$$

if and only if $\gamma$ itself has the property

$$\gamma = N_{t-e}(f_{t-e}), \qquad f_{t-e} \text{ in } Z_{t-e}.$$

* In his "Theory of Cyclic Algebras over an Algebraic Field," which will be published shortly in the *Transactions of the American Mathematical Society.*

† *Bulletin of the American Mathematical Society*, Vol. 37 (1931), pp. 301-312.

‡ See Theorem 5.

This implies that a cyclic algebra $A$ of degree $p^t$ has exponent $p^e$ (the only possible type of value as is known) if and only if

$$\gamma = N_{t-e}(f_{t-e}), \qquad \gamma \neq N_{t+1-e}(f_{t+1-e}).$$

In particular a construction of all cyclic normal division algebras over an algebraic field $\Omega$ is furnished * when it is shown that algebra $A$ of degree $p^t$ over $\Omega$ is a division algebra if and only if $\gamma$ is not the norm $N_1(f_1)$ of any quantity in the cyclic field $Z_1$ of order $p$ over $\Omega$.

2. *Elementary theorems.* We shall consider algebras over any non-modular field $F$. Let $M$ be the algebra of all $n$-rowed square matrices with elements in $F$. If $I$ is the $n$-rowed identity matrix and $x$ is any matrix of $M$ the equation

$$|\lambda I - x| = 0$$

is the characteristic equation $\phi(\lambda) = 0$ of $x$. Suppose that the equation $\phi(\lambda) = 0$ has no multiple roots. Then the invariant factors of $x$ are $I_n \equiv \phi(\lambda), 1, \cdots, 1$ since each invariant factor $I_j$ of $x$ is divisible by $I_{j-1}$ and $\phi(\lambda)$ is the product of the $I_j$. Hence *every* matrix $y$ with the same characteristic equation as $x$ has the same invariant factors as $x$ and, as is well known,† is a transform $zxz^{-1}$ of $x$ by a non-singular matrix $z$ of $M$.

Let $y$ in $M$ be commutative with $x$ in $M$. If the characteristic equation of $x$ has no multiple roots then $y$ is a polynomial ‡ in $x$ with coefficients in $F$. For if

$$\phi(\lambda) \equiv \lambda^n - \alpha_1\lambda^{n-1} - \cdots - \alpha_n = 0$$

is the characteristic equation of $x$ then $x$ is similar in $F$ to

$$\begin{Vmatrix} 0 & 0 & \cdots & 0 & \alpha_n \\ 1 & 0 & \cdots & 0 & \alpha_{n-1} \\ 0 & 1 & \cdots & 0 & \alpha_{n-2} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & \alpha_1 \end{Vmatrix}$$

and, without loss of generality, may be taken to have this form. Let $\xi_1, \cdots, \xi_n$ be the scalar roots of $\phi(\lambda) = 0$, all distinct by hypothesis. The Vandermonde matrix

---

* This result was proved by the author for $p = t = 2$ (*Bulletin of the American Mathematical Society, loc. cit.*).

† Cf. L. E. Dickson's *Modern Algebraic Theories*, p. 104.

‡ A well known result in matrix theory. The author gives the proof here for its novelty and because it shows the actual form of the polynomial.

$$V = \| \xi_i^{j-1} \|, \qquad (i, j = 1, \cdots, n),$$

is then non-singular. By an elementary computation

$$Vx = \xi V, \qquad \xi = \begin{Vmatrix} \xi_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \xi_n \end{Vmatrix}.$$

The matrix

$$t^{-1} = V'V = \| \sum_j \xi_j^{i+k-2} \| \qquad (i, k = 1, \cdots, n)$$

is non-singular and has elements in $F$. Let $\eta = VyV^{-1}$. Then

$$\eta = VyV^{-1} = VytV' = \| R(\xi_i, \xi_j) \| \qquad (i, j = 1, \cdots, n)$$

where if

$$yt = \| \beta_{ij} \|, \qquad (i, j = 1, \cdots, n),$$

then

$$R(\xi_i, \xi_j) = \sum_{s,t} \xi_i^{s-1} \beta_{st} \xi_j^{t-1} \qquad (i, j = 1, \cdots, n).$$

But $yx = xy$ if and only if $\xi\eta = \eta\xi$ so that

$$\| \xi_i R(\xi_i, \xi_j) - R(\xi_i, \xi_j)\xi_j \| = 0 \qquad (i, j = 1, \cdots, n).$$

Since $\xi_i = \xi_j$ only when $i = j$ we have $R(\xi_i, \xi_j) = 0$ $(i \neq j)$ and

$$\eta = \| R(\xi_i, \xi_i) \| = \begin{Vmatrix} P(\xi_i) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & P(\xi_n) \end{Vmatrix} = P(\xi)$$

where $P(\xi)$ has coefficients in $F$. But then $y = P(x)$.

Let $A$ be a normal simple algebra of order $n^2$ (degree $n$) over $F$. Algebra $M$ is a special case of $A$. As is well known $A = H \times D$ where $H$ is a total matric algebra and $D$ is a normal division algebra whose degree $m$ is called the *index* of $A$. If $\xi$ is any scalar root of the minimum equation of any quantity of grade $m$ in $D$ then, as was first proved by Wedderburn,*

$$A' = A \times F(\xi) = M \times F(\xi)$$

where $M$ is a total matric algebra. Algebra $A$ can then be thought of as an

* See the author's "On Direct Products," *Transactions of the American Mathematical Society*, Vol. 33 (1931), pp. 690-711, for reference and consequences of Wedderburn's result.

algebra over $F$ whose quantities are $n$-rowed square matrices with elements in $F(\xi)$.

The characteristic equation of the general matrix of $A$ is the so-called rank equation of $A$. If $x$ in $A$ has $\phi(\lambda) = 0$ of degree $n$ as its minimum equation with respect to $F$ then obviously $\phi(\lambda) = 0$ is the rank equation of $A$ for $x$ and is the characteristic equation of $x$. If moreover $\phi(\lambda) = 0$ has no multiple roots then, as we have shown, the only matrices of $A'$ commutative with $x$ are polynomials in $x$ with coefficients in $F(\xi)$. But then the only quantities of $A$ commutative with $x$ are polynomials in $x$ with coefficients in $F$. Let next $y$ in $A$ have the same minimum equation as $x$. Then $y = z_0 x z_0^{-1}$ where $z_0$ is in $A'$, as we have seen. We may write

$$z_0 = z_1 + z_2\xi + \cdots + z_m\xi^{m-1}$$

where the $z_i$ are in $A$. Since $z_0 x - y z_0$ is zero and $x$ and $y$ are in the algebra $A$ we have

$$\sum_i (z_i x - y z_i)\xi^{i-1} = 0$$

whence $z_i x = y z_i$. It follows that

$$(\sum_i z_i\xi_i)x = y(\sum_i z_i\xi_i)$$

for any scalars $\xi_i$. The matrix $\sum_i z_i\xi_i$ is non-singular for the values $\xi_1 = 1$, $\xi_2 = \xi, \cdots, \xi_m = \xi^{m-1}$ and hence the determinant of $\sum_i z_i\xi_i$ is not identically zero. It follows that there exist rational numbers $\eta_1, \cdots, \eta_m$ for which

$$z = z_1\eta_1 + z_2\eta_2 + \cdots + z_m\eta_m,$$

a quantity of $A$, is non-singular. Evidently $y = zxz^{-1}$, and we have proved

THEOREM 1. *Let $A$ be a normal simple algebra of order $n^2$ (degree $n$) over $F$ and let $\phi(\lambda) = 0$ of degree $n$ be the minimum equation of $x$ in $A$. Suppose that $\phi(\lambda) = 0$ has no multiple roots. Then the only quantities of $A$ commutative with $x$ are polynomials in $x$ with coefficients in $F$. Every quantity $y$ in $A$ with the same minimum equation as $x$ is a transform*

$$y = zxz^{-1}$$

*of $x$ by a non-singular quantity $z$ of $A$.*

Suppose that $M$ is a total matric algebra of degree $n$ over $F$. Algebra $M$ has an *ordinary matric basis.*

$$(e_{ij}) \qquad (i, j = 1, \cdots, n),$$

with a multiplication table

$$e_{ij} \cdot e_{jk} = e_{ik}, \qquad e_{ij} \cdot e_{tk} = 0 \qquad (j \neq t;\ i, j, t, k = 1, \cdots, n).$$

Let $y$ be in $M$ and have

$$\psi(\lambda) \equiv \lambda^n - 1 = 0$$

as its minimum equation for $F$. The scalar roots of $\psi(\lambda) = 0$ are the ordinary $n$-th roots of unity and are all distinct. Consider also the quantity

$$y_0 = e_{12} + e_{23} + \cdots + e_{n1}$$

a matrix which is actually the so-called *rational canonical form* of $y$, and hence, by Theorem 1, similar to $y$. There thus exists a non-singular quantity $z$ in $M$ such that $y_0 = zyz^{-1}$. Let

$$\epsilon_{ij} = ze_{ij}z^{-1},$$

so that the $(\epsilon_{ij})$ form a new ordinary matric basis of $M$. Then

$$y = z^{-1}y_0z = \epsilon_{12} + \epsilon_{23} + \cdots + \epsilon_{n1}.$$

THEOREM 2. *Let $y$ be a quantity of a total matric algebra $M$ of degree $n$ over $F$, and let*

$$\psi(\lambda) \equiv \lambda^n - 1 = 0$$

*be the minimum equation of $y$ with respect to $F$. Then $M$ has an ordinary matric basis*

$$(e_{ij}) \qquad (i, j = 1, \cdots, n)$$

*such that*

$$y = e_{12} + e_{23} + \cdots + e_{n1}.$$

3. *Cyclic algebras.* Let $Z = F(i)$ be a cyclic field of order $n$ over $F$ and let $\theta(i)$ in $Z$ be a root of the minimum equation $\phi(\lambda) = 0$ of $i$ for $F$ such that the $n$ distinct roots of $\phi(\lambda) = 0$ in $Z$ are $\theta^0(i) = \theta^n(i) = i$, $\theta(i)$, $\theta^2(i), \cdots, \theta^{n-1}(i)$. We may also represent the generator of the cyclic galois group of $\phi(i)$ by $\theta$. We define an algebra $A$ with the basis

$$(1) \qquad i^\alpha y^\beta \qquad (\alpha, \beta = 0, 1, \cdots, n-1),$$

and the multiplication table given by

$$(2) \qquad y^\beta i = \theta^\beta(i) y^\beta, \qquad y^n = \gamma \neq 0 \qquad (\beta = 0, 1, \cdots)$$

where $\gamma$ is in $F$. Algebra $A$ has been shown to be a normal simple algebra of degree $n$ over $F$ and is called a cyclic algebra.* We designate $A$ by

* In the section on cyclic algebras of my paper "On Direct Products," *Ibid.*

$$A = F(\gamma, Z, \theta). \tag{3}$$

Conversely every normal simple algebra $A$ of degree $n$ over $F$ which contains a cyclic subfield $Z$ of order $n$ over $F$ is a cyclic algebra, an immediate consequence of Theorem 1 for $y = \theta(i)$, $x = i$. The author has proved *

THEOREM 3. *The direct product*

$$F(\gamma_1, Z, \theta) \times F(\gamma_2, Z, \theta) = M \times F(\gamma_1\gamma_2, Z, \theta), \tag{4}$$

*where $M$ is a total matric algebra.*

Of particular importance in the theory of cyclic algebras is the structure of a cyclic algebra of degree $p^t$, $p$ a prime, where the algebra is

$$A = F(\gamma^p, Z, \theta).$$

The cyclic field $Z = Z_t$ contains a cyclic sub-field $Z_{t-1} = F(u)$ of order $q = p^{t-1}$, the field of all quantities of $Z$ symmetric in $\theta^q(i)$ and its $p-1$ iteratives. In fact $Z$ is a cyclic field of order $p$ over $Z_{t-1}$. Also

$$yuy^{-1} = u\,[\theta(i)] = \theta_{t-1}(u), \qquad y^q u = uy^q. \tag{5}$$

Let

$$z = \gamma^{-1}y^q \tag{6}$$

so that

$$z^p = \gamma^{-p}y^n = \gamma^{-p}\gamma^p = 1. \tag{7}$$

Algebra $A$ contains a cyclic sub-algebra $B$ of order $p^2$ over $F(u)$ with the basis

$$i^\alpha z^\beta \qquad (\alpha, \beta = 0, 1, \cdots, p-1) \tag{8}$$

and a multiplication table given by the minimum equation of $i$ with respect to $F(u)$ and

$$zi = \theta^q(i)z, \qquad z^p = 1. \tag{9}$$

But $B$ is a normal simple algebra of prime degree $p$ over $F(u)$ and is not a division algebra since $z^p = 1$ is impossible in a division algebra with a basis (8). Hence $B$ is a total matric algebra. Moreover $\psi(\lambda) \equiv \lambda^n - 1$ is evidently the minimum equation of $z$ with respect to $F(u)$ so that $B$ has an ordinary matrix basis

$$(e_{ij}) \qquad (i, j = 1, \cdots, n) \tag{10}$$

for $B$ such that, by Theorem 2,

$$z = e_{12} + e_{23} + \cdots + e_{n1}. \tag{11}$$

But then we may write

$$B = M_p \times F(u)$$

* In the section on cyclic algebras of my paper "On Direct Products," *Ibid.*

where $M_p$ is a total matric algebra with a basis (10) over $F$ and contains the quantity $z$ of (11).

By the well known Wedderburn theorem

$$A = M_p \times D \tag{12}$$

where $D$ is a normal simple algebra of degree $q = p^{t-1}$ over $F$. Algebra $D$ is the algebra of all quantities of $A$ commutative with all the quantities of $M_p$ and contains the cyclic field $Z_{t-1}$. It follows that $D$ is a cyclic algebra and has a basis

$$u^\alpha j^\beta \qquad (\alpha, \beta = 0, 1, \cdots, q-1), \tag{13}$$

where

$$j^\beta u = \theta^\beta_{t-1}(u) j^\beta, \qquad j^q = \delta \text{ in } F \qquad (\beta = 0, 1, \cdots). \tag{14}$$

But $yu = \theta_{t-1}(u)y$ so that $y^{-1}j$ is commutative with $u$.

The algebra of all quantities of $A$ commutative with $F(u)$ is evidently $B$ from the multiplication table (2) of $A$ (with $\gamma$ of course replaced by $\gamma^p$). Hence $y^{-1}j$ is in $B$. But since $j$ is in $D$ and $z$ is in $M_p$ the quantity $j$ is commutative with $z$. Also $y$ is commutative with $z = \gamma^{-1}y^q$. Hence $y^{-1}j$ in $B$ is commutative with $z$. The only quantities of $B$ commutative with $z$ are polynomials in $z$ with coefficients in $F(u)$ by (8) and (9). Hence $y^{-1}j = g(u, z)$, $j = yg = g[\theta_{t-1}(u), z]y = f(u, z)y$, where $f(u, z)$ is a polynomial in $z$ with coefficients in $F$. We record this result in the form of a lemma.

LEMMA. *The quantity $j$ of $D$ has the form*

$$j = f(u, z)y \tag{15}$$

*where $f(u, z)$ is a polynomial in $z$ with coefficients in $F(u)$.*

We now compute

$$\delta = j^q = N_{t-1}[f(u, z)]y^q \equiv f(u, z) \cdot f[\theta_{t-1}(u), z] \cdots f[\theta_{t-1}{}^{q-1}(u), z]y^q. \tag{16}$$

But $y^q = \gamma z$ so that (16) becomes

$$\delta\gamma^{-1} = N_{t-1}[f(u), z]z. \tag{17}$$

Let $\eta$ be an indeterminate and let

$$g(\eta) = N_{t-1}[f(u, \eta)]\eta, \tag{18}$$

a polynomial in $\eta$ with coefficients in $F$ and finite degree. We can evidently write

$$g(\eta) = \sum_{i=0}^{r} g_i(\eta)\eta^{p^i} \tag{19}$$

where the $g_i(\eta)$ are polynomials in $\eta$ of degree at most $p-1$ with coefficients in $F$. Also we write

$$g_i(\eta) = \sum_{j=0}^{p-1} g_{ij}(\eta) \qquad (g_{ij} \text{ in } F). \tag{20}$$

Since $z^p = 1$ we have

$$\gamma^{-1}\delta = g(z) = \sum_{i=0}^{r} g_i(z) = \sum_{j=0}^{p-1} \left(\sum_{i=0}^{r} g_{ij}\right) z^j. \tag{21}$$

But $1, z, \cdots, z^{p-1}$ are linearly independent with respect to $F$ so that

$$\sum_{i=0}^{r} g_{i0} = \delta\gamma^{-1}, \quad \sum_{i=0}^{r} g_{ij} = 0 \qquad (j = 1, \cdots, p-1). \tag{22}$$

We now consider the polynomial

$$f(u) = f(u, 1). \tag{23}$$

We have proved that

$$\begin{aligned} g(1) = N_{t-1}[f(u,1)] &= \sum_{i=0}^{r} g_i(1) \\ &= \sum_{j=0}^{p-1} \sum_{i=0}^{r} (g_{ij}) = \sum_{i=0}^{r} g_{i0} = \delta\gamma^{-1}. \end{aligned} \tag{24}$$

The quantity

$$j_0 = f(u)^{-1} j \tag{25}$$

differs from $j$ by a polynomial in $u$ with coefficients in $F$ as a left factor and we may replace $j$ in the basis of $D$ by $j_0$. But

$$j_0^q = N_{t-1}[f(u)]^{-1} j^q = \gamma\delta^{-1}\delta = \gamma. \tag{26}$$

We have now given a short proof of the fundamental result of the paper. The further important theorems on cyclic algebras will be easily derivable from this proved result.

THEOREM 4. *Let $A = F(\gamma^p, Z, \theta)$ be a cyclic algebra of degree $p^t$ over $F$, $p$ a prime, so that $Z$ contains a cyclic sub-field $Z_{t-1}$ of order $p^{t-1}$ over $F$ with $\theta$ as generator of its cyclic galois group. Then*

$$A = M_p \times F(\gamma, Z_{t-1}, \theta_{t-1}) \tag{27}$$

*where $M_p$ is a total matric algebra of degree $p$ over $F$.*

4. *The norm condition.* The author has proved

THEOREM 5. *Let*

$$n = p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m} \tag{28}$$

*where the $p_i$ are distinct primes. Every cyclic field $Z$ of order $n$ over $F$ and with $\theta$ as a generator of its galois group is a direct product*

$$Z = Z_1 \times Z_2 \times \cdots \times Z_m \tag{29}$$

*of cyclic fields $Z_i$ of orders $p_i^{b_i}$ respectively and with $\theta_i$ respectively as generators, and conversely. Every cyclic algebra*

$$A = F(\gamma, Z, \theta) \tag{30}$$

*of degree $n$ over $F$ is a direct product*

$$A = F(\gamma_1, Z_1, \theta_1) \times F(\gamma_2, Z_2, \theta_2) \times \cdots \times F(\gamma_m, Z_m, \theta_m), \tag{31}$$

*of cyclic algebras of degrees $p_i^{b_i}$ respectively and with the same $\gamma_i = \gamma$ as $A$. Conversely every direct product* (31) *is a cyclic algebra $F(\gamma, Z, \theta)$ whose $\gamma$ may be taken to be the $\gamma$ of all the $F(\gamma_i, Z_i, \theta_i)$. Algebra $A$ is a division algebra if and only if all the $F(\gamma_i, Z_i, \theta_i)$ are division algebras.*

The above theorem * reduces essentially all problems on cyclic algebras to the case where $n$ is a power of a prime $p$.

We shall now give a short proof of the theorem that $A$ $F(1, Z, \theta)$ is always a total matric algebra. By our proof of Theorem 4 this is certainly true when $n$ is a prime $p$. Let the theorem be true for algebras of degree $r < n$. If $n$ is not a power of a prime then Theorem 5 implies that $A$ is a direct product of total matric algebras and hence is a total matric algebra. Let then $n = p^t$, $p$ a prime. By Theorem 4 with $\gamma = 1$ we have $A = M_p \times F(1, Z_{t-1}, \theta_{t-1})$ is a direct product of total matric algebras and is a total matric algebra. The induction is complete and we have

THEOREM 6. *All cyclic algebras*

$$F(1, Z, \theta) \tag{32}$$

*are total matric algebras.*

We next give a short discussion of a necessary and sufficient condition that a general cyclic algebra be a total matric algebra. If $\gamma$ is the norm $N(f)$ of a quantity $f$ in $Z$ then by replacing $y$ in the basis (1) of $A$ by $f^{-1}y$ we replace $\gamma$ by $(f^{-1}y)^n = N(f^{-1})\gamma = 1$, so that $A = F(1, Z, \theta)$ is a total matric algebra. Conversely let $A = F(\gamma, Z, \theta)$ be a total matric algebra so that $A$ is equivalent to $F(1, Z, \theta)$. Then $A$ has a basis

$$i_0^{\alpha} y_0^{\beta} \qquad (\alpha, \beta = 0, 1, \cdots, n-1), \tag{33}$$

* For the results of this theorem see the author's "On Direct Products, Cyclic Division Algebras, and Pure Riemann Matrices," *Transactions of the American Mathematical Society*, Vol. 33 (1931), pp. 219-234.

such that

$$y_0 i_0 = \theta(i_0) y_0, \qquad y_0^n = 1. \tag{34}$$

But $i_0$ has the same minimum equation as $i$, a cyclic irreducible equation. By Theorem 1 $i_0 = ziz^{-1}$ where $z$ is in $A$. But then if

$$j = z^{-1} y_0 z \tag{35}$$

we have

$$j^n = 1, \qquad ji = \theta(i) j. \tag{36}$$

Since $yi = \theta(i) y$ the quantity $j^{-1}y$ is commutative with $i$ and is a polynomial in $i$ with coefficients in $F$. But then $y = f(i) j$ and $y^n = N(f) = \gamma$ with $f$ in $Z$.

THEOREM 7. *A cyclic algebra $F(\gamma, Z, \theta)$ is a total matric algebra if and only if $\gamma$ is the norm $N(f)$ of a quantity $f$ of $Z$.*

The exponent $\rho$ of a normal simple algebra $A$ over $F$ is defined to be the least integer for which $A^\rho$ is a total matric algebra. Let $A$ be a cyclic algebra of degree $n$. As the author has shown

$$A^\rho = M^{\rho-1} \times F(\gamma^\rho, Z, \theta), \tag{37}$$

where $M$ is a total matric algebra of degree $n$ (*On direct products, loc. cit.*). But then $F(\gamma^\rho, Z, \theta)$ is a total matric algebra so that, by Theorem 7,

$$\gamma^\rho = N(f), \tag{38}$$

where $f$ is in $Z$. Let $\sigma$ be the least integer such that $\gamma^\sigma$ is the norm of a quantity of $Z$. Then obviously $\sigma \leq \rho$. But $A^\sigma = M^{\sigma-1} \times F(\gamma^\sigma, Z, \theta)$ is a total matric algebra so that $\rho \leq \sigma$. Hence $\sigma = \rho$.

THEOREM 8. *The exponent $\rho$ of a cyclic algebra is the least integer $\sigma$ for which $\gamma^\sigma$ is the norm of a quantity of $Z$.*

We now let $n = p^t$, $p$ a prime. It is known that in this case $\rho$ is a power of $p$ (*On direct products, loc. cit.*), in fact a divisor of $p^t$. We thus have

THEOREM 9. *Let $Z$ be a cyclic field of order $p^t$ over $F$, $p$ a prime, and let $\gamma$ be in $F$. Then the least integer $\sigma$ for which $\gamma^\sigma$ is the norm of a quantity of $Z$ is a divisor of $p^t$.*

Let $\tau = p^s$, $\pi = p^{s-1}$ and suppose that $\gamma^\tau$ is the norm of a quantity of $Z$ while $\gamma^\pi$ is not such a norm. Then $\tau \geq \sigma$, the least integer for which $\gamma^\sigma$ is a norm. But if $\sigma = p^e$ then $e \leq s$. If $s = e + h$, $h > 0$ then $(\gamma^\sigma)^{p^{h-1}} = \gamma^{p^{e+h-1}} = \gamma^\pi$ is a norm, a contradiction. Hence $h = 0$ so that $\tau = \sigma$.

THEOREM 10. *If in Theorem 9* $\tau = p^s$, $\pi = p^{s-1}$ *such that* $\gamma^\tau$ *is the norm of a quantity of* $Z$ *while* $\gamma^\pi$ *is not such a norm then* $\sigma = \tau$.

We shall now prove

THEOREM 11. *Let* $Z = Z_t$ *be a cyclic field of order* $p^t$ *over* $F$, $p$ *a prime, so that we can define a sequence of cyclic fields*

$$Z_t, Z_{t-1}, \cdots, Z_i, \cdots, Z_1, Z_0 = F \tag{39}$$

*where* $Z_i$ *is a cyclic field of order* $p^i$ *over* $F$ *with* $\theta_i$ *as a generator of its cyclic group. We may then also define a sequence of relative norm symbols* $N_i(f_i)$ *where* $f_i$ *is in* $Z_i$ *and*

$$N_i(f_i) = f_i(u_i) \cdot f_i[\theta_i(u_i)] \cdots f_i[\theta_i^{p^i-1}(u_i)], \tag{40}$$

*is in* $F$. *A quantity* $\gamma$ *of* $F$ *has the property that*

$$\gamma^{p^e} = N_t(f_t) = N(f) \qquad f_t = f \text{ in } Z \tag{41}$$

*if and only if the quantity* $\gamma$ *itself has the property*

$$\gamma = N_{t-e}(f_{t-e}) \tag{42}$$

*for some* $f_{t-e}$ *in* $Z_{t-e}$.

We consider a cyclic algebra $A = F(\gamma, Z, \theta)$. Then if $\gamma^{p^e} = N(f)$ the algebra $A^{p^e}$ is a total matric algebra. But, by Theorem 4,

$$A^p = H_1 \times F(\gamma, Z_{t-1}, \theta_{t-1}), \tag{43}$$

where $H_1$ is a total matric algebra. Similarly

$$A^{p^2} = H_1{}^p \times [F(\gamma, Z_{t-1}, \theta_{t-1})]^p = H_2 \times F(\gamma, Z_{t-2}, \theta_{t-2}), \tag{44}$$

and in general

$$A^{p^e} = H_e \times F(\gamma, Z_{t-e}, \theta_{t-e}), \tag{45}$$

where $H_e$ is also a total matric algebra. But $A^{p^e}$ is a total matric algebra so that $F(\gamma, Z_{t-e}, \theta_{t-e})$ is a total matric algebra. By Theorem 7 equation (42) is satisfied.

Conversely let (42) be satisfied. Then obviously from the definition of the field $Z_{t-1}$ and similarly of the $Z_i$

$$\gamma^{p^e} = [N_{t-e}(f_{t-e})]^{p^e} = N_t(f_{t-e}),$$

so that our theorem is proved.

We shall now apply Theorems 9 and 10 to prove

THEOREM 12. *Let* $\gamma$ *be the norm of a quantity in* $Z_{t-e}$ *and not the*

*norm of a quantity of* $Z_{t+1-e}$. *Then the exponent of* $A = F(\gamma, Z, \theta)$ *of degree* $p^t$ *over* $F$ *is* $p^e$.

For if $\gamma$ is the norm of $f_{t-e}$ in $Z_{t-e}$ then, as we have seen $\gamma^{p^e} = N_t(f_{t-e})$. But also $\gamma^{p^{e-1}}$ is not the norm of a quantity of $Z$ since otherwise Theorem 11 would imply that $\gamma = N_{t+1-e}(f_{t+1-e})$, a contradiction of our hypothesis. Hence, if we put $\tau = p^e$, $\pi = p^{e-1}$ in Theorem 11, we obtain $\rho = \sigma = \tau$ as desired.

The consideration of $\gamma$ itself instead of a power of $\gamma$ is truly a remarkable simplification as working even with the case $p = t = 2$ has shown the author.

An immediate corollary of Theorem 12 and the fact that $\gamma$ is in $Z_0 = F$ and hence the norm $N_0(f_0)$ of a quantity $f_0 = \gamma$ of $F$, is

THEOREM 13. *A cyclic algebra* $A = F(\gamma, Z, \theta)$ *of degree* $p^t$ *over* $F$, $p$ *a prime, has exponent equal to its degree if and only if* $\gamma$ *is not the norm of any quantity of the cyclic field* $Z_1$ *of order* $p$ *over* $F$, *a subfield of* $Z$, *that is*

$$\gamma \neq N_1(f_1), \qquad f_1 \text{ in } Z_1.$$

Theorem 12 gave as simple a construction as is possible of algebras of degree $p^t$ and exponent $p^e$, an *intermediate* power of $p$, while Theorem 13 gives a construction for the exponent $p^t$, the other end value being given by Theorem 7. In Theorem 7 the algebras were total matric algebras while in Theorem 13 they are actually division algebras. For Wedderburn showed * that $A$ of degree $n$ is a division algebra if $\gamma^n$ is the least power of $\gamma$ which is the norm $N(f)$ of a quantity $f$ in $Z$, and hence, by Theorem 9, if $n$ is the exponent of $A$. A simpler proof is perhaps the following. If $A$ is a cyclic algebra $A = F(\gamma, Z, \theta)$ of degree and exponent $n$ so that $A = H \times D$ where $H$ is a total matric algebra of degree $s$ and $D$ is a normal division algebra of degree $t$ then $st = n$. But $A^t = H^t \times D^t$ is a total matric algebra since $H$ is a total matric algebra while, as is known (*On direct products, loc. cit.*) so also is $D^t$. Hence the exponent of $A$ is at most $t$. But the exponent of $A$ is $n$ so that $t = n$ and $A$ is a division algebra.

The above furnishes an essentially different proof of the Wedderburn norm condition. We have also proved.

THEOREM 14. *The algebras of Theorem* 13 *are cyclic normal division algebras over* $F$.

H. Hasse has considered cyclic algebras over algebraic number fields.

* Cf. L. E. Dickson's *Algebren und ihre Zahlentheorie*, p. 70, for Wedderburn's proof.

His algebraic theorems are evidently valid for any field $\Omega = R(\xi)$ where $\xi$ is *any* quantity satisfying an equation with rational coefficients and irreducible in $R$, the field of all rational numbers. Hasse has proved that the exponent of any cyclic division algebra over a field $\Omega$ is equal to its degree. But now Theorems 13 and 14 give

THEOREM 15. *A cyclic algebra* $A = (\gamma, Z, \theta)$ *of order* $p^{2t}$ *(degree* $p^t$*) over* $\Omega$*,* $p$ *a prime, is a division algebra if and only if*

$$\gamma \neq N_1(f_1)$$

*for any* $f_1$ *in the cyclic sub-field* $Z_1$ *of order* $p$ *over* $\Omega$*,* $Z_1$ *contained in* $Z$*.*

We have thus reduced the conditions for the construction of any cyclic normal division algebra over the most important type of algebra, to the condition *that a single quantity* $\gamma$ *be not a norm of any quantity in a cyclic field of prime order over* $\Omega$. Thus the general problem is of no more difficulty than the case where $n$ is a prime after the author's reduction has been made. This indeed furnishes a construction of all cyclic normal division algebras over a field $R(\xi)$.

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# ON SELLING'S METHOD OF REDUCTION FOR POSITIVE TERNARY QUADRATIC FORMS.

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1. *Introduction.* Two quadratic forms are said to be *equivalent* when one may be transformed into the other by a linear transformation with integral coefficients and of determinant unity. A *reduced form* is a form representing a *class* of equivalent forms. It is defined by means of inequalities on the coefficients so that every form is equivalent to a reduced form and no two reduced forms are equivalent to each other.

One set of inequalities first found by Seeber and simplified by Eisenstein define what we shall in this paper refer to as an *Eisenstein-reduced* form. The proof that a form so defined has the properties mentioned above is geometric and, though simplified enormously and put on a firm foundation by L. E. Dickson * is still somewhat complex. However, the resulting reduced form is a very desirable one in that the leading coefficient $a$ of the form is the least integer not zero represented by it and the second coefficient $b$ is the least integer not $ax^2$ represented by the form, where $x$ is an integer.

Selling † considers the form:

$$f = ax^2 + by^2 + cz^2 + 2gyz + 2hxz + 2kxy$$

and defines four new constants by the equations:

$$a + h + k + l = 0 \qquad b + g + k + m = 0$$
$$c + g + h + n = 0 \qquad d + l + m + n = 0$$

and calls $f$ reduced if $g, h, k, l, m, n$ are all $\leqq 0$. Charve ‡ establishes this definition by proving that these conditions define an $f$ uniquely except for permutations of $g, h, k, l, m, n$. Though Selling's proof followed geometric lines, Charve's is algebraic and this paper is based on the latter's work.

In view of the fact that permutations of $g, h, k, l, m, n$ alter materially the form $f$, it is necessary to add further inequalities if one is to select

* L. E. Dickson, *Studies in the Theory of Numbers*, pp. 155-180. See this also for references on the Eisenstein inequalities which are given in paragraph 3 of this paper.

† E. Selling, "Des formes quadratiques binaires et ternaires," *Journal de Mathématiques* (3), Vol. 3 (1877).

‡ L. Charve, "Réduction des formes quadratiques ternaires positives," *Annales Scientifiques de L'École Normale Supérieure* (2), Vol. 9 (1880).

one form $f$ from all those obtained by permutations of $g, h, k, l, m, n$. Then we will have a truly unique form. Borissow in his dissertation * considered this problem and constructed a table of reduced forms, according to Selling's definition, of determinant from 1 to 200. Though his supplementary inequalities were undoubtedly selected with a view to ease of computation, the resulting forms do not seem to be the best which can be obtained from Selling's inequalities. In view of the manifest advantages of the Eisenstein-reduced form it seems to the author highly desirable that the Selling reduced form should be made to correspond as closely with the Eisenstein reduced form as is consistent with Selling's original inequalities. That this is here accomplished is apparent from the discussion below.

The first part of this paper is devoted to the establishment, by algebraic methods based on those of Charve, of inequalities defining a unique reduced form and given in terms of the coefficients of $f$ in paragraph 3 of this paper.

It should clearly be noted that, while Selling's and Charve's discussion yields a form † $\phi$ (see paragraph 2) unique except for permutations of the coefficients, the form $f$ is *not* unique. For instance, the forms $x^2 + y^2 + z^2$ and $x^2 + y^2 + 3z^2 - 2xz - 2yz$ are both reduced under Selling's definition though they are equivalent. However, only the first of these forms satisfies inequalities (1), (2) and (3) of this paper. No two forms whose coefficients satisfy inequalities (1), (2) and (3) are equivalent unless they are identical, that is, unless corresponding coefficients are actually equal.

The third paragraph is devoted to a comparison of the above set and Eisenstein's set of defining inequalities. It is proved that every Eisenstein-reduced form for which $g, h$ and $k$ are not greater than zero is a Selling reduced form satisfying inequalities (1), (2) and (3) of paragraph 3—the closest correspondence possible, if Selling's original inequalities are to be retained. For the cases in which the forms do not coincide, a definite one-to-one correspondence is established between Eisenstein-reduced and Selling-reduced forms. This results in a stronger establishment of both theories, for the proof here contained may be thus considered as a proof of the Eisen-

* E. Borissow, "Reduction of Positive Ternary Quadratic Forms by Selling's Method, with a table of reduced forms for all determinants from 1 to 200," St. Petersburg 1890, 1-108 (Russian). The author here wishes to acknowledge his indebtedness to Professor J. V. Uspensky of Stanford University for the most generous loan of this book from his private library.

† Mordell, in *Proceedings of the Royal Society* (A), No. A 816, Vol. 131, p. 100, remarks that Selling found forms "leading to a *unique* reduced form." This is true only in the sense of the definition by Selling of a limited *class* of mutually equivalent forms any one of which may be considered to be the reduced form.

stein reduction inequalities or the proof of the Eisenstein reduction inequalities as a proof of the inequalities here.

In paragraph 4 the automorphs are listed.

2. *Supplementary inequalities.* We have associated with $f$ a form

$$\phi = ax^2 + by^2 + cz^2 + dt^2 + 2gyz + 2hxz + 2kxy + 2lxt + 2myt + 2nzt$$
$$= -g(y-z)^2 - h(z-x)^2 - k(x-y)^2 - l(x-t)^2 - m(y-t)^2 - n(z-t)^2.$$

As Charve points out, $\phi$ represents the same integers as $f$.

Selling calls $f$ reduced when $g, h, k, l, m, n$ are all $\leqq 0$. Charve establishes this by first proving that a necessary and sufficient condition that none of $g, h, k, l, m, n$ are positive is that

$$(a+b+c+d)/2 = -(g+h+k+l+m+n) = a+b+c+g+h+k$$

be a minimum under all equivalent transformations on $f$. Then, in order to show that $\phi$ is unique except for permutations of $g, h, k, l, m, n$ he defines two associated transformations

$$s = \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}$$

where $\alpha''' = -\alpha - \alpha' - \alpha''$, $\beta''' = -\beta - \beta' - \beta''$, $\gamma''' = -\gamma - \gamma' - \gamma''$,

$\alpha = \alpha_1 - \delta_1$, $\alpha' = \alpha_2 - \delta_2$, $\alpha'' = \alpha_3 - \delta_3$, $\alpha''' = \alpha_4 - \delta_4$, $\beta = \beta_1 - \delta_1$, etc.

The application of $S$ to $\phi$ is equivalent to the application of $s$ to $f$ and subtracting a number from all the elements of a column of $S$ leaves $s$ unaltered. We call $S$ of "determinant 1" when $s$ is of determinant 1.

For purposes of reference we state the following, which Charve proves,* as a lemma and reproduce a portion of his proof.

LEMMA. *Any transformation $S$, increases half the sum of the first four coefficients of $\phi$ by $-g(\sigma - 1)$ where*

$$2\sigma = (\beta_1 - \gamma_1)^2 + (\beta_2 - \gamma_2)^2 + (\beta_3 - \gamma_3)^2 + (\beta_4 - \gamma_4)^2$$

*and correspondingly for $h$, $k$, $l$, $m$, and $n$.*

If one denotes the coefficients of $\Phi$, the transformed $\phi$, by the corresponding capital letters we see $A = -g(\beta_1-\gamma_1)^2 - h(\gamma_1-\alpha_1)^2 - k(\alpha_1-\beta_1)^2 - l(\alpha_1-\delta_1)^2 - m(\beta_1-\delta_1)^2 - n(\gamma_1-\delta_1)^2$ and similarly for $B$, $C$ and $D$. Thus the coefficient of $-g$ in $A+B+C+D$ is $2\sigma$.

* L. Charve, *loc. cit.*, p. 15.

COROLLARY.* *If* $g \neq 0$ *the only transformations leaving* $(A+B+C+D)/2$ *unaltered in value,* i. e. *leaving the form reduced by Selling's definition, are those in which* $\sigma = 1$ *above,* i. e. *those for which two of* $(\beta_1 - \gamma_1)$, $(\beta_2 - \gamma_2)$, $(\beta_3 - \gamma_3)$, $(\beta_4 - \gamma_4)$ *are zero and the other two* $+1$ *and* $-1$ *respectively; similarly for the other coefficients.*

The proof is given in Charve's paper.

We find it convenient to define coefficients $a_{ij}$ as follows

$$a_{11} = a, \quad a_{22} = b, \quad a_{33} = c, \quad a_{44} = d;$$
$$b + 2g + c = a_{23} = a_{14} = a + 2l + d,$$
$$a + 2k + b = a_{12} = a_{34} = c + 2n + d,$$
$$a + 2h + c = a_{13} = a_{24} = b + 2m + d,$$
$$2g = b_{23}, \quad 2h = b_{13}, \quad 2k = b_{12}, \quad 2l = b_{14}, \quad 2n = b_{34}, \quad 2m = b_{24}.$$

Note $a_{ij} = a_{ji} = a_{kl}$ and $b_{ij} = b_{ji}$, where $i, j, k, l$ is a permutation of 1, 2, 3, 4. Note also that $a_{11} + a_{22} + a_{33} + a_{44} = a_{12} + a_{23} + a_{13}$.

Our first condition on a reduced form then is Selling's

$$(1) \qquad b_{ij} \leqq 0, \quad \text{i. e.} \quad a_{ii} + a_{jj} \geqq a_{ij} \qquad (i, j = 1, 2, 3, 4).$$

To select a unique form out of each set of 24 forms found by permuting $x, y, z, t$ we note that such a permutation merely permutes the subscripts of the $a$'s (and $b$'s) and the form is made unique by the following inequalities

$$(2.1) \qquad a_{ii} \leqq a_{jj} \quad \text{if} \quad i < j,$$

$$(2.2) \quad \text{if } a_{ii} = a_{jj} \text{ with } i < j \text{ then } a_{jk} \geqq a_{ik} \quad \text{i. e.} \quad |b_{ik}| \geqq |b_{jk}|$$

where $k$ is the lesser of 1, 2, 3, 4 not $i$ nor $j$.

This is easily seen if we note that the interchange of $i$ and $j$ in $a_{jk} > a_{ik}$ reverses the inequality and if $a_{jk} = a_{ik}$, i. e. $a_{il} = a_{jl}$, such an interchange leaves the form unaltered, i. e. corresponds to an automorph.

Consider the transformation $S_1$ which is a particular form of $S$

$$S_1 = \begin{pmatrix} \alpha_1 - \delta_1 & \alpha_2 - \delta_2 & \alpha_3 - \delta_3 & \alpha_4 - \delta_4 \\ \beta_1 - \delta_1 & \beta_2 - \delta_2 & \beta_3 - \delta_3 & \beta_4 - \delta_4 \\ \gamma_1 - \delta_1 & \gamma_2 - \delta_2 & \gamma_3 - \delta_3 & \gamma_4 - \delta_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that no three-rowed determinant taken from the first three rows of $S_1$ is zero if $s$ is non-singular for, from the definition of the $\alpha$'s, etc., $\alpha''' = l(\alpha, \alpha')$, $\beta''' = l(\beta, \beta')$, $\gamma''' = l(\gamma, \gamma')$ (where $l$ denotes a linear function) implies $\alpha'' = l'(\alpha, \alpha')$, $\beta'' = l'(\beta, \beta')$, $\gamma'' = l'(\gamma, \gamma')$. Thus it follows that no column of $S_1$ is a linear combination of two others if $s$ is non-singular.

* L. Charve, *loc. cit.*, p. 15.

Note also that $S$ has the same effect on $f$ and $\phi$ as the transformation $S'$ obtained from $S$ by multiplying each element of the matrix by $-1$. Thus if we are concerned merely with the resulting form we may use $S$ or $S'$ at pleasure, it being understood that if an explicit transformation $s$ on $f$ is desired we take that of $S$ or $S'$ for which $s$ is of determinant 1. Hereafter we thus call $S$ and $S'$ the "same" transformation.

THEOREM 1. *If* $ghklmn \neq 0$ *the inequalities* (1) *and* (2) *define a unique form and the only automorphs are permutations of* $x$, $y$, $z$ *and* $t$.

From the hypothesis and the corollary of the lemma it follows that every element of $S_1$ is 0, 1 or $-1$ and that every row as well as each set of differences between corresponding elements of two rows is $1, -1, 0, 0$ in some order. We thus have two possible types of $S_1$.

First, if three columns have three zeros and one column has one zero we see that each pair of columns having three zeros must have the element different from zero occurring in different rows. Consider two rows of the type

$$\begin{matrix} \epsilon & 0 & 0 & -\epsilon \\ 0 & \epsilon_1 & 0 & -\epsilon_1 \end{matrix}$$

the differences between corresponding elements being $\epsilon, -\epsilon_1, 0, -\epsilon + \epsilon_1$ showing $\epsilon_1 = \epsilon$ and thus, after permutation of columns we have

$$S_1 = \begin{pmatrix} \epsilon & 0 & 0 & -\epsilon \\ 0 & \epsilon & 0 & -\epsilon \\ 0 & 0 & \epsilon & -\epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } \epsilon = 1 \text{ or } -1.$$

Second, if two columns have two zeros and two have three, we note that two non-zero elements occurring in the same column are equal as above and thus, after permuting columns and multiplying each element by $-1$ if necessary we have as the three rows $1, -1, 0, 0$; $0, -1, 0, 1$; $0, 0, -1, 1$ and the differences between the elements between the first and third are $1, 1, -1, -1$ which is not allowable.

Thus the only transformations permissible here are those obtained by permuting columns of $S_1$ displayed above, i. e.

$$S = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$$

which is the identity transformation on $\phi$ or that obtained by multiplying

each element by $-1$. Thus the only permissible transformations merely permute $x$, $y$, $z$ and $t$ (or change the signs of all simultaneously). Such an interchange is barred by inequalities (2) unless it is an automorph and thus $f$ is unique. We denote such a transformation by $I$.

THEOREM 2a. *If $2h = b_{13} = 0$ and $gklmn \neq 0$ the only transformations leaving $(a+b+c+d)/2$ unaltered are transformations $I$ and those obtained by permuting the columns of*

$$T_{13} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The beginning sentence of the proof of theorem 1 holds here except that the difference between corresponding elements of the first and third rows need not be 0, 0, 1, $-1$ in some order. In fact, we will get new transformations only when it *is* not. However the sum of the differences must be zero. Under this hypothesis if we consider the last column to be that containing only one zero, $\alpha_4 - \delta_4 = \beta_4 - \delta_4 = \gamma_4 - \delta_4$ in the first division of the proof of theorem 1 and we see that we have no new transformation. The type excluded in the second division is precisely the type of $T_{13}$ if we note that interchanging the first and third rows of $T_{13}$ merely permutes the columns of $T_{13}$ and multiplies each element by $-1$.

THEOREM 2. *If $b_{ij} = 0$ for only one pair of values $i$, $j$, the only transformations leaving $(a+b+c+d)/2$ unaltered are transformations $I$ and those obtained by permuting the columns of $T_{ij}$, where $T_{ij}$ is obtained from $T_{13}$ by the following permutation of rows and columns*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & k & j & l \end{pmatrix}$$

*where $k < l$ and $i$, $j$, $k$, $l$ are 1, 2, 3, 4 in some order.*

This follows immediately from the above if we note that permuting the rows of any transformation $S$ permutes the variables in the original form and the permutation of the columns of $S$ permutes the variables in the resulting form.

Consider the following conditions on a form satisfying conditions (1) and (2)

(3) if $b_{ij} = 0$ then $|b_{ik}| \leqq a_{ii}$ and $|b_{jk}| \leqq a_{jj}$,
that is, if $a_{ii} + a_{jj} = a_{ij}$, then $a_{ik}, a_{jk} \geqq a_{kk}$

where $k$ is the lesser of 1, 2, 3, 4 not $i$ nor $j$ and $l$ is the greater.

We prove

THEOREM 3. *When only one of $g, h, k, l, m, n$ is zero, conditions* (1), (2) *and* (3) *determine a unique form to which every Selling reduced form is equivalent and $T_{ij}$ (defined above) is an automorph if and only if one of the three equivalent equalities: $a_{ik} = a_{kk}$, $a_{ii} = -b_{ik}$, $a_{ii} = -b_{il}$ hold. Transformations I and $T_{ij}$ are the only automorphs except possibly for certain permutations of the columns of $T_{ij}$.*

Under the transformation $T_{13}$ we have the following correspondence

$$\begin{array}{lccccccccc} f & a_{11} & a_{22} & a_{33} & a_{44} & a_{13} = a_{24} & a_{23} = a_{14} & a_{12} = a_{34} \\ f' & a'_{11} & a'_{12} = a'_{34} & a'_{33} & a'_{23} = a'_{14} & a'_{13} = a'_{24} & a'_{44} & a'_{22} \end{array}$$

and thus under $T_{ij}$ we have

$$\begin{array}{lccccccccc} f & a_{ii} & a_{kk} & a_{jj} & a_{ll} & a_{ij} = a_{kl} & a_{jk} = a_{il} & a_{ik} = a_{jl} \\ f' & a'_{ii} & a'_{ik} = a'_{jl} & a'_{jj} & a'_{kj} = a'_{il} & a'_{ij} = a'_{kl} & a'_{ll} & a'_{kk}. \end{array}$$

We consider $f$ to satisfy conditions (1) and (2) and thus have $a'_{ik} \leqq a'_{jk}$.

If $a_{ik} < a_{kk}$ we have $a'_{kk} < a'_{ik}$ which, with $a'_{ll} = a'_{ik} + a'_{jk} - a'_{kk}$ implies $a'_{ll} > a'_{kj} \leqq a'_{ik} > a'_{kk}$ which shows (3) is satisfied by $f'$, for variables corresponding to $k$ and $l$ are in their proper order.

If $a_{jk} < a_{kk}$ we have $a'_{ll} < a'_{ik} \leqq a'_{jk}$ which, as above, implies $a'_{jk} < a'_{kk}$ and thus $f'$ with $l$ and $k$ interchanged satisfies (3), for $a_{ik} = a_{jl}$ and $a_{jk} = a_{il}$.

If $a_{ik} = a_{kk}$ then $a_{ll} = a_{jk} \geqq a_{kk}$, $f$ satisfies (3) and $T_{ij}$ is an automorph. Note also that $a_{ii} = -b_{il} = -b_{ik}$.

If $a_{jk} = a_{kk}$ then $a_{ll} = a_{ik} \geqq a_{kk}$, $f$ satisfies (3) and $T_{ji}$ is an automorph.

If, finally, $a_{jk}$, $a_{ik} > a_{kk}$ we see that $a'_{ll}$, $a'_{kk} > a'_{ik} = a'_{jl}$ and thus $f'$ does not satisfy (3).

Thus, either $f$ or $f'$ but not both satisfies (1), (2) and (3) unless they are identical under a transformation $T_{ij}$. This completes the proof of theorem 3.

THEOREM 4a. *If $h = k = 0$,* (i. e. $b_{13} = b_{12} = 0$) *and $glmn \neq 0$ conditions* (1), (2), (3) *define a unique form $f$ and the only new automorphs are $K_{213}$ and possibly certain permutations of its columns where*

$$K_{213} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_{312}.$$

We first justify the notation if we note that permuting the second and third rows and the second and third columns of the transformation leaves it unaltered.

Now consider transformations $S_1$ where $1, -1, 0, 0$ are, in some order, the elements of each of the rows and the set of differences between corresponding elements of the second and third rows. These conditions must not hold for the differences between the first and second or first and third rows if we are to have a transformation not already considered.

First, if three columns have three zeros and one has one zero permute columns if necessary to have only one zero in the last column. Note that $\mp 1, \pm 1, \pm 1$ are the first three elements of the last column and we have $K_{213}$ above or the transformation obtained by multiplying each element by $-1$ or by permuting columns. Trial shows that $K_{213}$ is an automorph.

Second, if two columns have two zeros and two have three, permute columns if necessary to make the first row $1, 0, 0, -1$. Since $b_{23} \neq 0$ and no column has three elements different from 0, two non-zero elements in the third or second column of the second two rows must be identical. Permute columns if necessary to put them in the third column and have

$$\begin{matrix} 1 & 0 & 0 & -1 \\ \cdot & \cdot & \epsilon & \cdot \\ \cdot & \cdot & \epsilon & \cdot \end{matrix}$$

Then either the first or last columns but not both have three zeros. Permute the first and fourth columns and multiply by $-1$ if necessary to make the first column contain three zeros. Permute the second and third rows if necessary to get

$$\begin{matrix} 1 & 0 & 0 & -1 \\ 0 & -\epsilon & \epsilon & 0 \\ 0 & 0 & \epsilon & -\epsilon. \end{matrix}$$

Since the difference between the first and third rows is not $1, -1, 0, 0$ in some order we see $\epsilon = -1$ and we have that all such transformations are obtained by permuting the columns of the following two transformations

$$D_{123} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } D_{132} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, except for permutations of columns, $D_{123} = K_{213} \cdot T_{12}$ and $D_{132} = K_{213} \cdot T_{13}$ and, by the proof connected with conditions (3), $D_{123}$ and $D_{132}$ take $f$, satisfying (3), into a form which does not satisfy (3) unless $T_{12}$ or $T_{13}$ and therefore $D_{123}$ or $D_{132}$ are automorphs. This latter cannot happen because $T_{12}$, for example, is an automorph if and only if $-a_{11} = b_{13} = 0$ which is not true.

THEOREM 4. *If $b_{ij} = b_{ik} = 0$ and none of the other b's are 0, conditions (1), (2), (3) define a unique form f and the only new automorphs are certain permutations of the columns of $K_{jik}$ where $K_{jik}$ is obtained from $K_{213}$ of theorem 4a by permuting the rows and columns according to the plan*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$$

*and $K_{jik} = K_{kij}$.*

In the first place $K_{jik} = K_{kij}$ since the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & k & j & l \end{pmatrix} \text{ is } \begin{pmatrix} 1 & 3 & 2 & 4 \\ i & k & j & l \end{pmatrix} \text{ and } K_{312} = K_{213}.$$

Examining the changes made in any form $f$ by $K_{312}$ and noting the effect of the above permutation of 1, 2, 3, 4 we see that $K_{jik}$ has the following effect on the coefficients of any form $f$: leaves $b_{jk}$ unaltered, changes the sign of $b_{ik}$ and $b_{ij}$, replaces $b_{il}$ by $2b_{ik} + 2b_{ij} + b_{il}$, $b_{jl}$ by $2b_{ij} + b_{jl}$, $b_{kl}$ by $2b_{ik} + b_{kl}$ and increases the sum $(a + b + c + d)/2$ by $-(b_{ij} + b_{ik})$. $K_{jik}$ is thus an automorph when $b_{ij} = b_{ik} = 0$.

Now the only transformations coming under the heading of this theorem are those obtained from $K_{213}$, $D_{123}$ and $D_{132}$ by the permutations indicated. Since $T_{ij}$ increases the sum $(a + b + c + d)/2$ by $-b_{ij}/2$ for any form, it will increase the sum $(a + b + c + d)/2$ for the form resulting from the application of $K_{jik}$ by $b_{ij}/2 \leqq 0$. Thus the total effect of $K_{kij} \cdot T_{ij}$ on the sum $(a + b + c + d)/2$ of a form is to increase it by $-b_{ik} - b_{ij}/2$. Since $K_{kij} \cdot T_{ij}$ leaves $(a + b + c + d)/2$ unaltered when and only when $b_{ik} = b_{ij} = 0$ it must be $K_{jik}$, $D_{ijk}$ or $D_{ikj}$ (except perhaps for permutation of columns) and since of these three only $D_{ijk}$ increases $(a + b + c + d)/2$ by $-b_{ik} - b_{ij}/2$ (see the corollary to the lemma), we have the result that $K_{kij} \cdot T_{ij}$ must be obtained by a permutation of columns from $D_{ijk}$. Now, since $K_{kij}$ is an automorph of $f$ if $b_{ij} = 0 = b_{ik}$, the effect of $D_{ijk}$ on such a form is the same as that of $T_{ij}$. But, from the proof of theorem 2, $T_{ij}$ applied to $f$ satisfying (1), (2) and (3) yields a form not satisfying (3), $T_{ij}$ not being an automorph since $b_{ik} = 0 \neq a_{ii}$. Further, no permutation of variables in the resulting form can make it satisfy (1), (2) and (3). Thus $D_{ijk}$ and, similarly, $D_{ikj}$ can yield no new forms satisfying (1), (2) and (3); neither are they automorphs.

THEOREM 5a. *If $g = l = 0$ (i. e. $b_{23} = b_{14} = 0$) and $hkmn \neq 0$, conditions (1), (2) and (3) define a unique form f and there are no new automorphs.*

Here in $S_1$ the second and third rows and the differences between the corresponding elements of the first and second rows and of the first and third are 1, —1, 0, 0 in some order but this is not true of the first row or of the differences between corresponding elements of the second and third rows. The sum of the elements in any row is zero.

First, no element in the first row is $\pm 2$. For, if it were, the elements in the same column of the rows below it would be $\pm 1$ which implies that not more than one element of the first row is $\pm 2$. If there were such a single element we could multiply every element of $S_1$ by $-1$ if necessary to make 2 an element of the first row. Note that the other elements in the first row would then be $-1$, $-1$ and 0 and permute columns if necessary to have

$$\begin{array}{cccc} 2 & -1 & -1 & 0 \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{array}$$

At least one of the elements in the last column is $\neq 0$ and therefore is $-1$. If $-1$ is the element in the last column of the second row the differences between corresponding elements of the first and second rows would be 1, —1, —1, 1 which cannot be. The argument is the same for the third row.

Thus in the first row must occur two 1's and two —1's. Then, since $b_{12} \neq 0$ two of the elements in the second row must be directly below identical elements in the first. This holds for the third row also. Thus, after permutations of columns we have

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is singular. Thus results

THEOREM 5. *If $b_{ij} = b_{kl} = 0$ but none of the other b's are zero, conditions* (1), (2) *and* (3) *define a unique form f and there are no new automorphs.* (*i, j, k, l is a permutation of* 1, 2, 3, 4).

Note that $b_{ij} = b_{ik} = b_{jk} = 0$ implies no other $b$'s are zero for $b_{ij} = b_{ik} = 0$ implies $a_{ii} = -b_{il} \neq 0$, etc.

THEOREM 6a. *If $g = h = k = 0$, conditions* (1), (2) *and* (3) *uniquely determine f and there are no new automorphs.*

Any of the first three rows of $S_1$ has 1, —1, 0, 0 in some order, as its

elements but this is not true of the sets of differences between corresponding elements of any pair of rows from the first three. It can be seen from the proof of theorem 4a that the only case to consider here is that in which two columns have two zeros and two have three. Permute columns if necessary to make two non-zero elements occur in each of the first and third columns. Two elements in the same column must be of opposite sign and thus, by permuting the first three rows and multiplying each element by $-1$ if necessary we have for the first and third columns 1, $-1$, 0, 0 and 0, 1, $-1$, 0. Permute the second and fourth columns if necessary to make the element $-1$ of the first row occur in the second column, note that the last column cannot be all zeros and that

$$L_{123} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the only new transformations leaving $(a+b+c+d)/2$ unaltered are those obtained from $L_{123}$ by permuting its columns and the first three rows. We note that interchanging the first and third rows of $L_{123}$ merely permutes the columns and thus all transformations here to be considered are obtained by permutations of the columns of $L_{123}$, $L_{132}$ and $L_{213}$.

Now, for any form $f$, $L_{123}$ increases the sum $(a+b+c+d)/2$ by $-(b_{12}+b_{23}+b_{13}/2)$ and $L_{ikj}$ increases it by $-(b_{ik}+b_{jk}+b_{ij}/2)$. It may be seen from the proof of theorem 4 that, for any form $f$, $K_{ikj}$ increases $(a+b+c+d)/2$ by $-(b_{ik}+b_{jk})$ and leaves $b_{ij}$ unaltered. Thus $K_{ikj} \cdot T_{ij}$ increases $(a+b+c+d)/2$ by $-(b_{ik}+b_{jk}+b_{ij}/2)$. Thus $K_{ikj} \cdot T_{ij}$ being a transformation leaving $(a+b+c+d)/2$ unaltered when and only when $b_{ik}=b_{jk}=b_{ij}=0$, must be $L_{ikj}$ or obtained from it be a permutation of columns. We have three different transformations of this type for permutations of $i, j, k$. Now $K_{ikj}$ is an automorph for $f$ if $b_{ij}=b_{ik}=b_{jk}=0$ and thus $L_{ikj}$ has the same effect as $T_{ij}$. Thus, if $f$ satisfies conditions (3) as well as (1) and (2), its transformation by $L_{ikj}$ does not satisfy (3). $L_{ikj}$ cannot be an automorph because $T_{ij}$ is an automorph only if $-b_{il}=-b_{ik}=a_{ii}$ and $b_{ik}=0$ here.

This gives

THEOREM 6. *If $b_{ij}=b_{ik}=b_{jk}=0$, conditions (1), (2) and (3) uniquely determine $f$ and there are no new automorphs.*

Finally we prove

THEOREM 7. *Every positive ternary quadratic form is equivalent to a unique form f satisfying conditions* (1), (2) *and* (3).

This will follow from the theorems above if we prove

$$b_{ij} = b_{ik} = b_{jl} = 0 \text{ is inconsistent with (1), (2), (3).}$$

First note that it results from the hypothesis that

$$2a_{ii} = -b_{il},\ 2a_{jj} = -b_{jk} \text{ and thus } b_{il} \cdot b_{jk} \neq 0,$$
$$2a_{kk} = -b_{jk} - b_{kl},\ 2a_{ll} = -b_{il} - b_{kl} \text{ and thus } b_{kl} \neq 0$$

for, if it were, $\phi$ would be expressible as $a_{ii}v^2 + a_{jj}u^2$ where $v$ and $u$ are each the difference of a pair of variables from $x$, $y$, $z$ and $t$, which would mean that the determinant of $f$ would be zero. Now, if $f$ is to satisfy (1) and (2) we see that $i < l$ and $j < k$. Also $a_{kk} \leqq a_{ll}$ if and only if $-b_{jk} = 2a_{jj} \leqq -b_{il} = 2a_{ii}$. Noting that $b_{ij} = b_{ik} = b_{jl} = 0$ remains unaltered by the permutation $(ij)(kl)$ we see that the only values for $i$, $j$, $k$ and $l$ which need be considered are 1, 2, 4, 3 and 1, 3, 4, 2. In the first case $k = l = g = 0$, i. e. $a = -h$ which contradicts (3) with $i = 1$, $j = 2$. In the second case $h = l = g = 0$, i. e. $a = -k$ which contradicts (3) with $i = 1$, $j = 3$.

3. *A comparison of Selling- and Eisenstein-reduced forms*

$$f = ax^2 + by^2 + cz^2 + 2gyz + 2hxz + 2kxy.$$

We extend the Selling conditions for a reduced form henceforth in this paper by calling a form *Selling-reduced* if it satisfies not only conditions (1) but also (2) and (3). The form is unique from theorem 7. Translated into terms of the coefficients of $f$ alone we have the following conditions for a Selling-reduced form $f$

(1.1) $$g, h, k \leqq 0,$$

(1.2) $$a \geqq |h + k|\ ;\quad b \geqq |g + k|\ ;\quad c \geqq |g + h|\ ^*$$

(2.1) $$a \leqq b \leqq c;\quad a + b + 2g + 2h + 2k = d - c \geqq 0.$$

(2.2) $$\text{If } a = b,\ |g| \leqq |h|\ ;\quad \text{if } b = c,\ |h| \leqq |k|\ ;$$
$$\text{if } a + b + 2g + 2h + 2k = 0,\ a \leqq |2h + k|\ .$$

(3.1) $$\text{If } h = 0,\ a \geqq 2|k| \text{ and } c \geqq 2|g|\ ;$$
$$\text{if } k = 0,\ a \geqq 2|h| \text{ and } b \geqq 2|g|\ ;$$
$$\text{if } g = 0,\ b \geqq 2|k| \text{ and } c \geqq 2|h|\ .$$

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* The last condition is redundant in view of the inequalities below.

(3.21) If $a = -h-k$, then $a \geqq 2\,|\,k\,|$ and $a+c+2h \geqq b$,
i.e. $|\,h\,| \geqq |\,k\,|$ and $c+h \geqq b+k$.

(3.22) If $b = -g-k$, then $b \geqq 2\,|\,k\,|$ and $b+c+2g \geqq a$,
i.e. $|\,g\,| \geqq |\,k\,|$ and $c+g \geqq a+k$.

(3.23) If $c = -g-h$, then $c \geqq 2\,|\,h\,|$ and $b+c+2g \geqq a$,
i.e. $|\,g\,| \geqq |\,h\,|$ and $b+g \geqq a+h$.

We list the Eisenstein conditions for a reduced form *

(A) $g, h, k$ all $\leqq 0$ or all $> 0$.

(B) $a \geqq 2\,|\,h\,|,\ a \geqq 2\,|\,k\,|,\ b \geqq 2\,|\,g\,|$.

(C1) $a \leqq b \leqq c,\ [a+b+2g+2h+2k \geqq 0]$.†

(C2) If $a = b$, $|\,g\,| \leqq |\,h\,|$; if $b = c$, $|\,h\,| \leqq |\,k\,|$;
[if $a+b+2g+2h+2k = 0$, $a \leqq |\,2h+k\,|$].†

(D1) For $g, h, k \leqq 0$: if $a = -2k$, $h = 0$;
if $a = -2h$, $k = 0$; if $b = -2g$, $k = 0$.

(D2) For $g, h, k > 0$: if $a = 2k$, $h \leqq 2g$;
if $a = 2h$, $k \leqq 2g$; if $b = 2g$, $k \leqq 2h$.

We prove the following lemmas relating the reduced forms of the two types.

LEMMA 1. *Every Eisenstein-reduced form with $g, h, k \leqq 0$ is Selling-reduced and only such are Selling-reduced.*

Consider such an Eisenstein-reduced form. Then, obviously all the conditions for a Selling-reduced form are satisfied if we note that (B) and (C1) imply $b \geqq 2\,|\,k\,|,\ c \geqq 2\,|\,g\,|,\ c \geqq 2\,|\,h\,|$. No form with $g, h$ and $k > 0$ can be a Selling-reduced form.

LEMMA 2. *A Selling-reduced form is Eisenstein-reduced if two or more of $g, h, k$ are 0 or if none of the following mutually exclusive conditions are satisfied:*

I. $a < -2h$ *or* $a = -2h$ *and* $k \neq 0$.
II. $a < -2k$ or $a = -2k \neq -2h \neq 0$.
III. $b < -2g$ or $b = -2g$ *and* $k \neq 0$.

Inspection of the conditions shows that a Selling-reduced form is Eisen-

* L. E. Dickson, *loc. cit.*

† The condition in brackets is omitted if $g, h, k$ are all $> 0$.

stein-reduced unless it fails to satisfy one of (B) or (D1). If $h = k = 0$ condition (3.1) shows that conditions (B) are satisfied and similarly for $g = h = 0$ and $g = k = 0$. I, II, III clearly include all cases in which one of (B) or (D1) is denied. To show that the three conditions are mutually exclusive note that I, (2.13) and (1.21) imply $b \geqq -2g - 2k \geqq -2g$, $b = -2g$ only if $k = 0$, $a \geqq -2k$ with the equality holding only if $a = -2h$. The proof is similar for II. Finally III and (2.13) implies $a \geqq -2h - 2k$, $a$ not being equal to $-2k$ unless $h = 0$ nor to $-2h$ unless $k = 0$.

LEMMA 3. *Let $f'$ be a form with $a < -2h$ or $a = -2h$ and $k \neq 0$ and satisfying* (1) *and one of the following mutually exclusive sets of conditions:*

I'. $a \leqq b \leqq c$, *the first equality holding only if* $|g| \leqq |h|$, *the second equality not holding unless* $a = -2h = -2k$; *also* (2.13), (2.2), (3.1), (3.21).

II'. $a \leqq c \leqq b$, $h \neq k$, *the first equality holding only if* $|g| \leqq |k|$; $a + c + 2g + 2h + 2k \geqq 0$, *the equality holding only if* $a \leqq |2k + h|$; (3.1).

III'. $b < a \leqq c$, *the equality holding only if* $|g| \leqq |k|$; (2.13), (3.1) *and* (3.21) *the equality in* (2.13) *not holding.*

*One and only one form $f'$ may be obtained by a permutation of variables of a form $f$ which is Selling-reduced but not Eisenstein-reduced. Conversely, one and only one form $f$, Selling-reduced but not Eisenstein-reduced, may be obtained by a permutation of variables of a form $f'$.*

We first prove that for no form $f'$ is $b = -g - k$ or $c = -g - h$ possible. Note that in each of I', II', III' the sum of each pair of elements $a$, $b$, $c$ is not less than $-(2g + 2h + 2k)$. This, with $a \leqq -2h$ implies $b \geqq -2g - 2k > -g - k$. Also $a + c + 2h \geqq -2g - 2k$ and, since in each of I', II', III' is $c \geqq a$, $a + c + 2h \geqq 2a + 2h$. These, together, imply that $a + c + 2h \geqq a + h - g - k$, i. e. $c \geqq -h - g - k$. Thus $c > -g - h$ unless $k = 0$ which implies $a \geqq -2h$, $k = 0$, from (3.1), contrary to the conditions on $f'$.

If a form $f$ occurs in I of lemma 2, its coefficients satisfy the inequalities of *I'*. $b \neq c$ from (2.22) unless $a = -2h = -2k$ since $a \leqq -2h$ and $a \geqq -h - k$ implies $|k| \leqq |h|$, the last equality holding only if the two previous ones hold. Thus, in view of the paragraph above, $f$ in I and $f'$ in I' are identical. Furthermore, any permutation of variables alters $a \leqq -2h$

unless $a = -2h = -2k$ and the permutation is $(yz)$. But $(yz)$ alters $b \leqq c$ unless $b = c$ in which case it is an automorph.

If a form $f$ occurs in category II permute $y$ and $z$ in $f$. This does not affect (1), converts II into I with $a \neq -2k$ if $a = -2h$, makes the changes noted in (2), the equality $c = b$ not considered since $|h| > |k|$ in any case, and (3.1) is left unaltered. Furthermore $a \neq -h-k$ from (3.21) since, before the permutation of $y$ and $z$, $a < -2k$ or $a = -2k \neq -2h$. Interchange of $y$ and $z$ in conditions (3.22) and (3.23) interchanges the initial conditions $b = -g-k$, $c = -g-h$ and thus, from the first paragraph of this proof, none of (3.2) need be considered. Thus the conditions on $f'$ in II′ are equivalent to the conditions II and (1), (2), (3) on $f$. Further permutation of the variables of $f'$ alters the inequalities as above and our unique correspondence is established for this case.

If a form $f$ occurs in category III note that (2.13) implies $a \geqq -2h-2k$, the equality holding only if $k \neq 0$ and therefore, from (2.23), $a+b+2g+2h+2k \neq 0$. Also $a \neq b$ since $|g| > |h|$. Now permute $x$ and $y$ leaving (1) unaltered and making the indicated alterations in (2). (3.1) is left unaltered by the interchange and (3.22) becomes (3.21). As, in the previous paragraph, no other (3.2) need be taken into account. The correspondence is thus established as above.

I′, II′, III′ are categorical and mutually exclusive since I, II and III are.

LEMMA 4. *In every Eisenstein-reduced form*

$$F = Ax^2 + By^2 + Cz^2 + 2Gyz + 2Hxz + 2Kxy$$

*with $G, H, K > 0$, the variables $x, y, z$ may be permuted to give a unique form $F'$ with $B - 2G \geqq A - 2H$, $H \geqq K \leqq G$ the first equality holding only if $A \leqq B$, the second only if $C \geqq B$, the third only if $C \geqq A$ and the last two only if $A \leqq B \leqq C$. Furthermore this form will satisfy conditions* (B) *and* (D2) *together with those obtained by permuting $x$, $y$ and $z$ in the conditions. Every such form will fall into one of the three following mutually exclusive categories:*

I″. $A \leqq B \leqq A + C - 2H$, *the second equality not holding unless* $A = 2H = 2K$ *and* $B = C$.

II″. $A \leqq A + C - 2H \leqq B$, $A \neq H + K$.

III″. $B < A \leqq A + C - 2H$.

*Furthermore no permutations of variables of the form $F$ will yield two dis-*

*tinct forms falling into the same or different categories and one and only one form $F$ may be obtained by a permutation of variables of a form $F'$.*

Note that for every Eisenstein-reduced form with $H, K, G > 0$ not only do (B) and (D2) hold but also those conditions obtained from (B) and (D2) by permuting $x, y, z$ with the corresponding change of designation of coefficients.

Consider a form $F$, Eisenstein-reduced with $G, H, K > 0$. Suppose one of $G, H, K$ is less than the other two. Interchange variables if necessary until $H > K < G$. Then, without altering $H > K < G$, one may permute $x$ and $y$ if necessary to have a form for which $B - 2G > A - 2H$ or $B - 2G = A - 2H$ with $A \leqq B$.

If two of $G, H, K$ are equal and less than the third we may permute variables to have a form for which $G = K < H$, the permutation being thereby uniquely defined except for an interchange of $x$ and $z$.

a) If $B - 2G \geqq A - 2H$ and $C \geqq A$, we have it in the form desired.

b) If $B - 2G \geqq A - 2H$ and $C < A$, note $B - 2G = B - 2K > C - 2H$ and an interchange of $x$ and $z$ makes $B - 2G > A - 2H$ and $C > A$.

c) If $B - 2G < A - 2H$ but $B - 2G \geqq C - 2H$ note $C < A$, interchange $x$ and $z$ having $B - 2G \geqq A - 2H$ with $C > A$.

d) If, in any of the above cases, after the permutations noted we have $B - 2G = A - 2H$, $C \geqq A$ with $A > B$, interchange $x$ and $y$ to get $B - 2G = A - 2H$, $B > A$, $C \geqq B > A$ and $K = H < G$ in terms of the coefficients of the new form.

e) If $B - 2G < A - 2H$, $C - 2H$ interchange $x$ and $y$ and have $K = H < G$, $A - 2H < B - 2G$, $C - 2G$. These inequalities remain unaltered by a permutation of $y$ and $z$ and thus we can make $C \geqq B$.

If $G = H = K$, we may permute $x, y$ and $z$ so that $A \leqq B \leqq C$ and see that then $B - 2G \geqq A - 2H$. Thus we have shown that a permutation giving the required inequalities exists.

In order to show that this permutation is unique except for automorphs (i. e. the form is uniquely defined) suppose the coefficients of a form satisfy the requirements of lemma 4 (without I″ II″, III″). The interchange of $x$ and $y$ alters $B - 2G \geqq A - 2H$ unless $B - 2G = A - 2H$ and then alters $B \geqq A$ unless $A = B$ when $G = H$ and the form itself is unaltered. Similarly an interchange of $x$ and $z$ alters $K \leqq G$ unless $K = G$ and then $C \geqq A$ unless $C = A$; similarly for the interchange of $y$ and $z$. The permutations

$(xyz)$ and $(xzy)$ leave $H \geqq K \leqq G$ unaltered if and only if $H = K = G$ in which case they alter $C \geqq B \geqq A$ unless $C = B = A$. If $A = B = C$ and $H = K = G$ both permutations are automorphs.

Finally, noting that, even after permutation of variables, $C \geqq 2H$ and therefore $A \leqq A + C - 2H$, we see that I″, II″, III″ form a set of categorical and mutually exclusive relationships between $A$, $B$ and $A + C - 2H$. (In II″ $A \neq H + K$ for $A = H + K$ if and only if $A = 2H = 2K$, $C \geqq B$, $C \leqq B$, and therefore $C = B$, which is included in I″.)

THEOREM 8. *Every form $f$ satisfying the conditions of lemma* 3 *is equivalent to a unique form $F$ satisfying the conditions of lemma* 4 *and conversely. Furthermore, $f$ occurs in category* I′ *if and only if $F$ occurs in* I″ *and similarly for* II′ *and* II″ *as well as* III′ *and* III″.

First, suppose we have a form satisfying the conditions of lemma 3. Transform this form $f$ by

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and get $f' = ax^2 + by^2 + (a + c + 2h)z^2 - 2(g + k)yz + 2(a + h)xz - 2kxy$. Call this form $F$, denoting the coefficients by the corresponding capital letters. We note that if there is any correspondence by this transformation it must be I′ — I″, etc. It remains to prove that $f' = F$ satisfies the remaining conditions on $F$ in lemma 4 together with (A), (D2), (B) and the conditions obtained from (B) and (D2) by permutations of the variables.

Conditions (A): $k \neq 0$ for $k = 0$ implies by (3.1) that $a \geqq -2h$. Thus $-g - k \neq 0$ and $a + h \neq 0$ since $a \geqq -h - k$.

Conditions (B) and those obtained from them by permuting $x$, $y$, $z$: $a \geqq 2(a + h)$ since $a \leqq -2h$; $a \geqq -2k$ similarly; $b \geqq -2g - 2k$ since $a \leqq -2h$ and since, for all of I′, II′, III′, (2.13) holds; therefore $b \geqq -2k$. Also $a + c + 2h \geqq -2g - 2k$ from (2.13); $a + c + 2h \geqq 2a + 2h$ since $c \geqq a$.

Conditions (D2) and those obtained from them by permuting $x$, $y$, $z$: If $a = -2k$ it is obvious that $a + h \leqq -2g - 2k$; similarly if $a = 2a + 2h$, $-k \leqq -2g - 2k$. If $b = -2g - 2k$, it follows from (1.2) that $-k \leqq 2a + 2h$. $b = -2k$ is impossible unless $g = 0$ for, from (2.1), $a \geqq -2h - 2g$; then $-g - k \leqq 2(a + h)$ from (1). If $a + c + 2h = 2a + 2h$, i.e. $c = a$, then $-g - k \leqq -2k$, i.e. $-g \leqq -k$ follows from the conditions for the equality $c = a$ in I′, II′ and III′. If $a + c + 2h = -2g - 2k$, it follows that $a + h \leqq -2k$ from the first equality and

(2. 1) which imply that $b$ is greater than or equal to the greater of $a$ and $c$, which can happen only in II′ and when $a \leqq -2k-h$.

The conditions of lemma 4: (2. 13) implies $B-2G \geqq A-2H$. Suppose the equality held with $A > B$, i. e. $a > b$. This latter occurs only in III′; but in that case the equality of (2. 13) does not hold. In the second place, $K \leqq G$, i. e. $-k \leqq -g-k$ is true, the equality holding only if $g=0$ thus implying by (3. 13) that $c \geqq -2h$ and therefore $a+c+2h = C \geqq A$. Thirdly, $K \leqq H$, i. e. $-k \leqq a+h$ by (1. 21), the equality holding only if $a=-h-k$ and therefore when $C \geqq B$. Finally $K=H=G$ implies $g=0$, $a=-h-k$ and thus from (2. 13), $b \geqq -h-k=a$ and thus $B \geqq A$ and $C \geqq B$ from the above.

Second, suppose we have the form $F$ satisfying the conditions of lemma 4 together with (A), (B), (D2) and those obtained from (B) and (D2) by permutation of $x, y, z$. It is clear that, in virtue of (C), (B) and (D2) imply the conditions obtained from them by permutation of $x, y, z$ and therefore, below, in citing these conditions we shall include the conditions so obtained from them.

Transform $F$ by

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(the inverse of the previous transformation) and get

$$F' = Ax^2+By^2+(A+C-2H)z^2-2(G-K)yz-2(A-H)xz-2Kxy.$$

We prove $F'=f$ satisfies the conditions I′, II′ or III′ and $a<-2h$ or $a=-2h$ with $k \neq 0$ and that $f$ satisfies (1).

$a \leqq -2h$ the equality holding only when $k \neq 0$ if and only if $A \leqq 2A-2H$, i. e. $2H \leqq A$, $K \neq 0$ which follows from the conditions given.

(1. 1) hold, for $G-K \geqq 0$, $A-H>0$, $K>0$.

(1. 2): $A \geqq A-H+K$, i. e. $H \geqq K$; $B > G-K+K=G$; $A+C-2H > G-K+A-H$, i. e. $C-H > G-K$.

(2. 13): for all permutations it is true that $A+B$, $2A+C-2H$, $B+A+C-2H \geqq 2G+2A-2H$. This is true since $B-2G \geqq A-2H$. The equality can hold only if $B-2G=A-2H$ or II″ and $C=2G$.

(2. 21) and (2. 22) for I′: if $A=B$, $G-K \leqq A-H$, i. e. $A \geqq G+H-K$ since $H \geqq G$ and $A \geqq 2H$; if $B=A+C-2H$, $A-H \leqq K$ for the first equality by I″ implies $A=2H=2K$.

(2. 21) and (2. 22) for II′: if $A=A+C-2H$, $G-K \leqq K$ is true since the first implies $C=2H$ and (D2) applies. $K \neq A-H$ is true.

(2.22) for III′: $A = A + C - 2H$, $G - K \leqq K$ as above.

(2.23): for I′ we have the equality only if $B - 2G = A - 2H$; then $A \leqq 2A - 2H + K$ since $A \geqq 2H - K$. For II′ we have the equality only if $C = 2G$; then $A \leqq 2K + A - H$, i. e. $H \leqq 2K$ from (D2).

(3.1): The only possible equality in (1.1) above is $G = K$, i. e. $g = 0$. Then $B \geqq 2K$ and $A + C - 2H \geqq 2A - 2H$, i. e. $C \geqq A$, the latter being true in virtue of lemma 4 with $G = K$.

(3.21): For I′ the only equality of (1.2) is $H = K$ above. $A \geqq 2K$ and $a + c + 2h = C \geqq B$ is true from the above. For III′ with $H = K$ we have the same.

To complete the proof call $T$ the transformation of the second part of the proof of this theorem. $T$ takes a form $F$ satisfying the conditions of lemma 4 into a form $f$ satisfying the conditions of lemma 3. Suppose that there is another equivalent transformation $T'$ (that is, with integral coefficients and of determinant unity) taking $F$ into $f' \neq f$ satisfying the conditions of lemma 3. Then $T^{-1}$ would take $f'$ into $F' \neq F$ where $F'$ satisfies the conditions of lemma 4. Then $f' \sim f$ and $F' \sim F$. By lemma 4, $F'$ and $F$ are equivalent to two forms $F'_1$ and $F_1$, $F'_1 \neq F_1$, which are Eisenstein-reduced. By lemma 3, $f'$ and $f$ are equivalent to two forms $f'_1$ and $f_1$, $f'_1 \neq f_1$, which are Selling-reduced. This occurrence contradicts the proof that no two Eisenstein-reduced forms are equivalent as well as the corresponding proof for Selling-reduced forms.

Thus we have also proved the

COROLLARY. *No two Selling-reduced forms are equivalent if and only if no two Eisenstein-reduced forms are equivalent.*

Thus the two theories of reduction are correlated.

It is interesting to note here that Charve proves (*op. cit.*, p. 18) that the three least integers * not necessarily distinct represented by a Selling-reduced form are included among the following:

$$a,\ b,\ c,\ a + b + c + 2g + 2h + 2k,\ a + b + 2k,\ a + c + 2h,\ b + c + 2g.$$

Since $a \leqq b \leqq c \leqq a + b + c + 2g + 2h + 2k$ we may exclude the fourth quantity in the list.

4. *Automorphs.* The automorphs of forms satisfying Eisenstein's inequalities are found in *Studies in the Theory of Numbers* (*op. cit.*). We accordingly find here merely the automorphs for those forms whose coefficients satisfy inequalities (1), (2) and (3) but not all of (B) and (D1), i. e. which occur in one of the categories I, II, III of the last section.

* Exclusive of $x^2a$, $ax^2 + by^2 + 2bzxy$, etc.

The ten cases on page 180 of *Studies* in which none of $g$, $h$, $k$ (that is, $r$, $s$, $t$) are zero correspond to the automorphs which are permutations of $x$, $y$, $z$ and the fourth variable $t$. It may be quickly verified that of these only the following can occur for a Selling reduced form [that is, satisfying (1), (2) and (3)] which is not Eisenstein-reduced:

| | Conditions on the form | Automorphs |
|---|---|---|
| $1_1$ | $a+2h+k=b+2g+k=0$ | $(z-x,\ z-y,\ z)$ |
| $1_2$ | $a=b,\ g=h$ | $(-y,\ -x,\ -z)$ |
| $1_3$ | $b=c,\ h=k$ | $(-x,\ -z,\ -y)$ |
| $1_4$ | $a=b,\ a+b+2g+2h+2k=N=0$ | $(y-z,\ x-z,\ -z)$ |
| $1_5$ | $a=b=c,\ N=0$ | $(z-y,\ -y,\ x-y)$ |
| | | $(-x,\ z-x,\ y-x)$ |
| $1_6$ | $a=b=c,\ N=0,\ g=h$ | $(x-z,\ x,\ x-y)$ |
| | | $(y,\ y-z,\ y-x)$ |
| $1_7$* | $a=b=c,\ N=0,\ h=k$ | $(y-z,\ y-x,\ y)$ |
| | | $(x,\ x-y,\ x-z)$ |
| | | $(z-y,\ z,\ z-x).$ |

If one and only one of $g$, $h$, $k$, $l$, $m$, $n$ is zero we prove that we have to consider merely the following five conditions on the form and the corresponding automorphs:

| | Conditions on the form | Automorphs |
|---|---|---|
| $2_1$ | $b=-2k,\ g=0\neq h$ | $(-x,\ y-x,\ -z)$ |
| $2_2$ | $c=-2h,\ g=0\neq k$ | $(-x,\ -y,\ z-x)$ |
| $2_3$ | $c=-2g,\ h=0\neq k$ (thus $a\neq b$) | $(-x,\ -y,\ z-y)$ |
| $2_4$ | $a=-2k=-2h$ | $(x-y-z,\ -y,\ -z)$ |
| $2_5$ | $a=-h-k,\ b+k=c+h$ | $(y+z-x,\ z,\ y).$ |

*Proof.* Consider the cases of theorem 3. The three cases $a=-2k$, $h=0$; $a=-2h$, $k=0$; $b=-2g$, $k=0$ occur only when the form is Eisenstein-reduced since $a=-2k$ implies $b\geqq -2g$ and $b=-2g$ implies from (2. 2) that $a\leqq -k$ which is false, etc. $b=-2k=-2g$ implies, from (2. 1) that $a\geqq -2g-2h>b$ unless $h=0$ and $a=-2g=b$ when, from (2. 2), $|g|\leqq 0$ which is false. The same argument holds for $c=-2h=-2g$. Also $b=-g-k$ implies $a\geqq -g-k-2h>b$ unless $h=0$; and a similar argument holds for $c=-g-h$. The form of the automorph for each remaining case can be obtained from theorem 3.

We have already shown (lemma 2) that a Selling-reduced form is

* This can occur only in combination (iv) below.

Eisenstein reduced if two of $g$, $h$, $k$ are zero. Suppose $b_{ij} = b_{ik} = 0$ where one of $i$, $j$, $k$ is 4 and none of the other $b$'s, except perhaps $b_{jk}$, is zero. Then $2a_{ii} = -b_{il}$, $2a_{ll} = -b_{il} - b_{kl} - b_{jl} > 2a_{ii}$. Thus $l > i \neq 4$. Thus we have to consider only $b_{ij} = b_{i4} = 0$. Note that, from theorem 3, $|b_{ik}| \leqq a_{ii}$ and $|b_{jk}| \leqq a_{jj}$ implies $|b_{il}| \geqq a_{ii}$ and $|b_{jl}| \geqq a_{jj}$ when $b_{ij} = 0$. Then $|b_{i4}| \geqq a_{ii}$ where $b_{i4} = 0$, which is impossible.

Thus, except for the identity, the only automorphs for any form are those above and, in certain cases, those obtained by certain permutations of the columns of automorphs $2_1$ to $2_5$. Investigation reveals that for any form Selling-reduced but not Eisenstein-reduced, one of three things may occur: first, none of the conditions $2_1$ to $2_5$ hold in which case the automorphs are given by $1_1$ to $1_6$; second, none of the conditions $1_1$ to $1_6$ hold and not more than one of $2_1$ to $2_5$ hold, in which case the automorphs are as given; third, one of the combinations below holds.

| | Conditions on the form | Automorphs |
|---|---|---|
| (i) | $a = -2h = -2k$, $b = c$ | $1_3$, $2_4$, $2_5$. ($2_4 \cdot 1_3 = 2_5$ and $2_5 \cdot 1_3 = 2_4$) |
| (ii) | $a = b = -h - k$, $c = -2h$, $g = 0$ | $1_4$, $2_2$, $2_5$, $2_2 \cdot 1_4$, $2_5 \cdot 1_4$. |
| (iii) | $a = b = -2k = -2h$, $g = 0$ | $1_4$, $2_1$, $2_4$, $2_1 \cdot 1_4$, $2_4 \cdot 1_4$. |
| (iv) | $a = b = c = -2k = -2h = 2$, $g = 0$ | See below. |

In the last case above which includes only one primitive form transform the form by the first transformation of lemma 5 to get an Eisenstein-reduced form. Its automorphs are given in *Studies*. Note that no pair of (i), (ii), (iii) can hold without (iv) holding.

We give the following multiplication table for the combinations above:

| $A =$ | $2_1$ | $2_2$ | $2_4$ | $2_5$ |
|---|---|---|---|---|
| $A \cdot 1_4 =$ | $(z-y, x-y, z)$ | $(z-y, z-x, -y)$ | $(y+z-x, z-x, z)$ | $(x-y-z, -z, x-$ |

From the above information the automorphs of any form may be found if it is Selling-reduced but not Eisenstein-reduced.

# A NEW TYPE OF ARITHMETICAL INVARIANCE.

By E. T. Bell.

1. The arithmetical invariance in question will be more easily seen from an example than from formal definitions. The following notation is fixed.

$m \equiv$ an arbitrary constant odd integer $> 0$.

$\mu$, $\nu$, $\delta$, $\tau$, with suffixes, are variable integers; the $\mu$, $\delta$, $\tau$ are odd, the $\nu$ odd or even; the $\delta$, $\tau$ are $> 0$, the $\mu \geqq 0$, and the $\nu \gtreqless 0$.

$$\Phi_{si} \equiv \sum_{a=1}^{2(m^2-1)-4s+4i} \mu^2_{sia} + 4 \sum_{b=1}^{2(m^2-1)-4i} \nu^2_{sib} + 2 \sum_{c=1}^{2s+1} \delta_{sic}\tau_{sic},$$

$$(s = 0, \cdots, (m^2-1)/2; \quad i = 0, \cdots, s).$$

A sum $\sum_p^q$ in which $q < p$ is vacuous and is to be ignored.

$g(z)$ is uniform and finite for all integer values of $z$; $g(z) = -g(-z)$; $g(0) = 0$; $g(z)$ is otherwise arbitrary.

Each of $e_1, \cdots, e_{2s+1}$ is a definite one of $+1$ or $-1$, and $\sum_e$ refers to the $2^{2s+1}$ values of the matrix $(e_1, \cdots, e_{2s+1})$.

$N$ is an arbitrary integer $\geqq 0$.

$A_{si}(m)$, $B_{si}(m)$ are integers, and depend upon $m$ alone.

The invariance considered refers to the $A_{si}(m)$, $B_{si}(m)$ (in this example), and concerns functions $g(z)$ whose arguments $z$ are certain linear, homogeneous functions of all the $\tau$'s obtained from the set of all representations of $N$ simultaneously in the system of $(m^2+1)(m^2+3)/8$ forms $\Phi_{si}$ defined above. Each form $\Phi_{si}$ is in $2(2m^2-1)$ indeterminates $\mu$, $\nu$, $\delta$, $\tau$. The total number of $A_{si}(m)$, $B_{si}(m)$ in this *invariant set* for the system of forms $\Phi_{si}$ is $(m^2+1)(m^2+3)/4$. The invariance is of the kind that the linear relation between the functions $g(z)$ just described has as its coefficients the integers in the invariant set, *and this relation is the same for all $N$ represented in the system of forms $\Phi_{si}$.* The trivial invariant set in which each $A_{si}(m)$, $B_{si}(m)$ is zero is excluded; *there is precisely one non-trivial invariant set.* The same invariant set appertains to an infinity of distinct systems of forms. We proceed to state the linear relation and to indicate how the corresponding invariant set may be found.

With respect to the form $\Phi_{si}$, for fixed $(s, i)$, we construct first the sums

$$g_{sim}(N) \equiv \sum g\Big(\sum_{c=1}^{2s} e_c \tau_{sic} + m e_{2s+1} \tau_{si\,2s+1}\Big),$$

$$g_{si1}(N) \equiv \sum g\Big(\sum_{c=1}^{2s} e_c \tau_{sic} + e_{2s+1} \tau_{si\,2s+1}\Big),$$

where $\sum$ refers to all solutions of $\Phi_{si} = N$, and $(e_1, \cdots, e_{2s+1})\,(s \geqq 0)$ is the general matrix of $e$'s. Write

$$g_{si}^{(m)}(N) \equiv \sum_e e_1 \cdots e_{2s+1}[A_{si}(m) g_{sim}(N) + B_{si}(m) g_{si1}(N)],$$

where $\sum_e$ (as stated above) refers to the $2^{2s+1}$ $e$-matrices, and where the $A$, $B$ are for the moment arbitrary constants. Thus $g_{si}^{(m)}(N)$ is defined for $\Phi_{si}$, $(s, i)$ fixed. Finally we sum $g_{si}^{(m)}(N)$ over all forms $\Phi_{si}$ in the system of $(m^2+1)(m^2+3)/8$.

THEOREM. *There is precisely one non-trivial* ($\equiv$ *not all zero*) *invariant set of* $(m^2+1)(m^2+3)/4$ *integers*

$$A_{si}(m), B_{si}(m) \qquad (s = 0, \cdots, (m^2-1)/2;\ i = 0, \cdots, s)$$

*such that*

$$\sum_{s=0}^{(m^2-1)/2} \sum_{i=0}^{s} g_{si}^{(m)}(N)$$

*vanishes for all integers* $N \geqq 0$; *the integers in the invariant set do not depend upon* $N$.

As already remarked, the *invariance* consists in the fact that the unique invariant set is the same for all $N \geqq 0$. The following features of such theorems are noteworthy: the *uniqueness* of the invariant set; the fact that the number of indeterminates in each form of the system concerned is of the *second degree* in the given constant integer (here $m$), namely $4m^2 - 2$ here; the arbitrariness of the function $g$; the existence of *an infinity* of systems of forms having the same invariant set.

The above set is based on an arbitrary *odd* positive integer $m$, and an arbitrary *odd* function $g(z)$, for the particular system of forms $\Phi_{si}$. Invariant sets also exist for even $m$, or for unrestricted $m$, and for even functions $g$ or for completely arbitrary functions.

The distinctive feature of such invariant sets is the *form of the number of indeterminates*, namely, *a polynomial of degree* 2 in the *given integer* ($m$ above).

2. The origin of the invariant set in § 1 can be briefly indicated. Write

$$\lambda(x) \equiv \Phi_2 \vartheta_3 \vartheta_1(x)/\vartheta_0(x), = 4 \sum q^{t/2} [\sum \sin \delta x],$$

where the theta notation is that of Jacobi, and the first $\sum$ refers to all odd integers $t > 0$, the second to all positive divisors, $\delta$, of $t$. Then, for $m$ an odd integer $> 0$, as in § 1, we have

$$\lambda(mx)/\lambda(x) = G_1^{(m)}(w)/G_0^{(m)}(w),$$

where $w \equiv \lambda^2(x)$, and the $G$'s are polynomials in $w$, $k^2$, where $k$ is the modulus of $\lambda(x)$,

$$G_1^{(m)}(z) = m + \sum_{s=1}^{(m^2-1)/2} a_{1s}^{(m)} w^s,$$

$$G_0^{(m)}(z) = 1 + \sum_{s=1}^{(m^2-1)/2} a_{0s}^{(m)} w^s,$$

in which $a_{js}^{(m)}$ $(j = 0, 1)$ is a polynomial in $k^2$ of degree at highest $t - 1$ in $k^2$, with rational integer coefficients. Assuming these polynomials known, we use $k^2 = \vartheta_2^4/\vartheta_3^4$, and proceed to paraphrase the resulting identity.*

As similar reductions have been explained in detail in the places cited, and elsewhere, we shall omit them, especially as they add nothing to the main object of this note, which is to call attention to a remarkable new type of arithmetical invariance.

Considerably more may be said about the integers in the invariant set of the Theorem than is there stated, and similarly for all such sets. In a paper to be published elsewhere I have shown how these integers can be given explicitly as polynomials in $m$.

The uniqueness of the invariant set is an immediate consequence of the like (easily proved by a simple contradiction, if not already classic) for the formulas giving the real multiplication of elliptic functions.

The infinity of systems of forms having the same invariant set follows by paraphrasing any of the identities, equivalent to that obtained from $\lambda(mx)/\lambda(x)$, by introducing any common factor, composed of thetas or

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* See, for example, *Transactions of the American Mathematical Society*, Vol. 22 (1921), p. 1; or "Algebraic Arithmetic," *Colloquium Publications of the American Mathematical Society*, Vol. 7 (1927). In the last refer particularly to p. 48, (10. 4), for the appropriate $e$-theorem concerning the separation of the products of sines, arising from the powers of $\lambda(x)$, into the corresponding sums. From these we write down the integers $A$, $B$ of the Theorem in § 1, as $\pm$ the respective integer coefficients in the polynomials $a_{js}^{(m)}$.

elliptic functions, into the numerator and denominator of the expression for $\lambda(mx)/\lambda(x)$. At any stage the elliptic functions can be replaced by their equivalents as theta quotients, to give another system of forms appertaining to the same invariant set.

That the number of indeterminates is a quadratic polynomial in the given integer ($m$ in the Theorem), follows from the multiplication of elliptic functions at once, and is obvious from the classic transcendental solution of the problem of real multiplication.

Finally, it is obvious that invariant sets arise from the real multiplication of *any* elliptic function. The like applies to the wider theory of transformation.

# THE ARITHMETIC OF POLYNOMIALS IN A GALOIS FIELD.*

By LEONARD CARLITZ.†

## INTRODUCTION.

In this paper we consider some of the arithmetic properties of polynomials in an indeterminate, $x$, with coefficients lying in some (finite) Galois field. The totality of such polynomials forms a ring in which arithmetic appears to be simpler than that of the ordinary integers; on the other hand, it is by no means a trivial arithmetic. What treatment of the subject appears in the literature does not seem to sufficiently stress the inherent simplicity of the polynomial domain: theorems are proved by utilizing the analogy with the rational domain. An instance of this is Artin's ‡ proof of Dedekind's § Law of Quadratic Reciprocity.

The first section of this paper is concerned with the treatment of the simplest numerical functions connected with the domain under consideration. An interesting feature is that the analogs of many difficult asymptotic formulas of ordinary arithmetic are here capable of simple explicit expression. A well-known case is the expression for the number of (primary) irreducible polynomials of a given degree. We give a number of others of the same type. For example, if $\tau(F)$ denote the number of primary divisors of the polynomial $F$, then

$$\sum \tau(F) = (\nu + 1)p^{\pi\nu},$$

the summation extending over the primary polynomials $F$ of degree $\nu$; again, if $Q(\nu)$ denote the number of primary "quadratfrei" polynomials of degree $\nu$, then

$$Q(\nu) = p^{\pi\nu} - p^{\pi(\nu-1)}.$$

The method used in proving these and similar results is not elementary in that it depends on equating coefficients in equal power series. However it appears to be the natural one for the subject, and there seems to be little point to recasting the proofs in "arithmetic" shape.

* Presented to the American Mathematical Society, December 30, 1930.

† National Research Fellow.

‡ E. Artin, "Quadratische Körper im Gebiete der höheren Kongruenzen," *Mathematische Zeitschrift*, Vol. 19 (1924), pp. 153-246.

§ R. Dedekind, "Abriss einer Theorie der höheren Congruenzen in Bezug auf einer reellen Primzahl-Modulus," *Journal für die reine und angewandte Mathematik*, Vol. 54 (1857), pp. 1-26.

In the second section a theorem of reciprocity is set up which includes Dedekind's quadratic law as a special case. If the underlying Galois field be of order $p^\pi$, we define the residue character of index $p^\pi - 1$

$$\left\{\frac{A}{P}\right\} \equiv A^{(p^{\pi\nu}-1)/(p^\pi-1)}, \bmod P;$$

then if $P$, $Q$ are primary and irreducible of respective degrees $\nu$, $\rho$, we prove that

$$\left\{\frac{P}{Q}\right\} = (-1)^{\rho\nu} \left\{\frac{Q}{P}\right\}.$$

From this we may of course deduce reciprocity of index any divisor of $p^\pi - 1$; on the other hand, to treat the case in which the index is not a divisor of $p^\pi - 1$ we have merely to sufficiently enlarge the polynomial domain by passing to a larger Galois field.

This theorem of reciprocity probably best indicates the comparatively simple nature of the ring of polynomials as opposed to ordinary arithmetic—where reciprocity of index higher than the second necessitates the consideration of algebraic fields. The proof here given is quite elementary and superficially resembles one of Kronecker's proofs of the ordinary quadratic reciprocity theorem.

## 1. Numerical Functions.

1. *Notation.* Take $p$ any prime (including 2) and $\pi$ any positive integer. Let $GF(p^\pi)$ denote the unique Galois field * of order $p^\pi$. If $x$ be an indeterminate, $\mathfrak{D}(p^\pi, x)$ will denote the totality of polynomials in $x$ with coefficients lying in the $GF(p^\pi)$. Evidently $\mathfrak{D}(p^\pi, x)$ is contained in $\mathfrak{D}(p^{\pi_1}, x)$ if and only if $\pi_1$ is a multiple of $\pi$. We shall find it convenient to employ the following notation. The elements (polynomials) of $\mathfrak{D}$ will be denoted by large Roman letters, rational integers will be denoted by small Greek and Roman letters, but small Roman $a$ will be reserved for the elements of the Galois field. $A = B$ will of course mean that the coefficients of like powers of $x$ are identical elements of the underlying Galois field. Further, if

$$F = a_0 x^\nu + a_1 x^{\nu-1} + \cdots + a_\nu,$$

then

$$\operatorname{sgn} F = a_0, \quad |F| = p^{\pi\nu};$$

$F$ is *primary* if $\operatorname{sgn} F = 1$. If $F = aG$, then $F$ and $G$ are *associates.*

* The properties of Galois fields used here are developed in Dickson's *Linear Groups* (1901), pp. 3-54.

Evidently the number of primary polynomials of degree $\nu$ is $p^{\pi\nu}$. Hence the $\zeta$-function in $\mathfrak{D}$,

$$(1) \qquad \zeta(s) = \sum_{\operatorname{sgn} F=1} 1/|F|^s = \sum_{\nu=0}^{\infty} p^{\pi\nu}/p^{\pi\nu s} = 1/[1-1/p^{\pi(s-1)}]$$

Also since decomposition of the elements of $\mathfrak{D}$ into irreducible polynomials is essentially unique we have the Euler factorization

$$(2) \qquad \zeta(s) = \sum 1/|F|^s = \prod 1/(1-1/|P|^s),$$

the product extending over all primary *irreducible* polynomials $P$ in $\mathfrak{D}$; both series and product converge absolutely for $s > 1$.

From (2) we may immediately derive the well-known expression for $\Psi(\nu)$, the number of (primary) irreducible polynomials of degree $\nu$: Indeed * from (1) and (2) follows

$$1-1/p^{\pi(s-1)} = \prod_{\operatorname{sgn} P=1} (1-1/|P|^s)$$

$$= \prod_{\nu=1}^{\infty} (1-1/p^{\pi\nu s})^{\psi(\nu)},$$

so that by taking logarithms

$$\sum_{\mu,\nu=1}^{\infty} \psi(\nu)/\mu p^{\pi\mu\nu s} = \sum_{\nu=1}^{\infty} 1/\nu p^{\pi\nu(s-1)},$$

whence †

$$\sum_{\delta|\nu} \delta\psi(\delta) = p^{\pi\nu},$$

which is equivalent to

$$(3) \qquad \psi(\nu) = (1/\nu)(p^{\pi\nu} - \sum p^{\pi\nu/\omega_1} + \sum p^{\pi\nu/\omega_1\omega_2} - \cdots),$$

$\omega_1, \omega_2, \cdots$, denoting distinct primes dividing $\nu$.

We shall suppose in the following that all summations or products over a subset $\mathfrak{S}$ of $\mathfrak{D}$ extend only over *primary* polynomials in $\mathfrak{S}$ unless the contrary be stated.

2. *Definitions.* As numerical functions of primary interest we define $\mu, \lambda, \tau, \sigma, \phi, Q$.

(i) $\mu(F)$, the analog of the Möbius-$\mu$, is defined by

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* This proof for $\pi = 1$ is given by Landau in an editorial note on a paper by H. Kornblum, "Ueber die Primfunktionen in einer arithmetischen Progression," *Mathmatische Zeitschrift*, Vol. 5 (1919), p. 107.

† For $\delta \mid \nu$ read, as usual, $\delta$ divides $\nu$.

$$\begin{cases} \mu(a)=1,\ \mu(aF)=\mu(F), \\ \mu(F)=0 \text{ for } P^2 \mid F, \\ \mu(F)=(-1)^\rho \text{ for } F=aP_1\cdots P_\rho\ (P_i\neq P_j). \end{cases}$$

(ii) $\lambda(F)$, the analog of the Liouville function, is defined by

$$\lambda(a)=1, \qquad \lambda(aF)=\lambda(F),$$
$$\lambda(F)=(-1)^{a_1+\cdots+a_\rho} \quad \text{for} \quad F=aP_1^{a_1}\cdots P_\rho^{a_\rho}.$$

(iii) If $\operatorname{sgn} F=1$, $\tau(F)$ is the number of primary divisors of $F$:

$$\tau(F)=\sum_{D\mid F} 1,$$

extending over primary $D$ only. More generally, $\tau_\rho(F)$ is the number of solutions of $F=D_1D_2\cdots D_{\rho+1}$ ($\operatorname{sgn} D_i=1$), so that $\tau(F)=\tau_1(F)$.

(iv) $$\sigma(F)=\sum_{D\mid F} \mid D \mid ;$$

$\sigma_\rho(F)=\sum_{D\mid F} \mid D \mid^\rho$, so that $\sigma(F)=\sigma_1(F)$.

(v) After Dedekind, $\phi(F)$ is the number of elements in a reduced residue system, mod $F$. We define $\phi_\nu(F)$ to be the number of primary polynomials of degree $\nu$ that are prime to $F$, whence it is easy to see that

$$\phi_\nu(F)=\phi(F) \quad \text{if} \quad \deg F=\nu.$$

(vi) $Q(\nu)$ will denote the number of primary polynomials of degree $\nu$ not divisible by the square of an irreducible polynomial. More generally $Q_\rho(\nu)$ will denote the number of primary polynomials of degree $\nu$ not divisible by the $(\rho+1)$-th power of an irreducible polynomial; $Q_1(\nu)=Q(\nu)$.

3. *The $\lambda$- and $\mu$-functions.* We note first that, exactly as in the rational case, $\mu$ appears in inversions of sums (or products) extended over the divisors of a fixed element. If $f(F)$ be any function in $\mathfrak{D}$ then ($\operatorname{sgn} F=1$)

$$f'(F)=\sum_{D\mid F} f(D) \quad \text{and} \quad f(F)=\sum_{D\mid F}\mu(D)f'\left(\frac{F}{D}\right) \tag{4}$$

are equivalent. This follows at once from the identity

$$\Sigma f'(F)/\mid F\mid^s=\Sigma f(F)/\mid F\mid^s\cdot\Sigma 1/\mid F\mid^s.$$

However we may derive more interesting relations for both $\mu$ and $\lambda$—relations analogous to the asymptotic formulas connected with these functions in the rational domain. Since

$$\sum_F \mu(F)/\mid F\mid^s=\prod_P(1-1/\mid P\mid^s)=1/\zeta(s)=1-p^\pi/p^{\pi s},$$

we have immediately

$$(5)\qquad \begin{cases} \sum\limits_{\deg F=\nu} \mu(F) = 0 \quad \text{for} \quad \nu \geqq 2, \\ \sum\limits_{\deg F=1} \mu(F) = -p^{\pi}. \end{cases}$$

Similarly from

$$\sum_F \lambda(F)/|F|^s = \prod_P 1/(1+1/|P|^s)$$

$$= \prod (1-1/|P|^s)/(1-1/|P|^{2s}) = \zeta(2s)/\zeta(s)$$

$$= \sum_{\nu=0}^{\infty} p^{\pi\nu}/p^{2\pi\nu s}(1-p^{\pi}/p^{\pi s}),$$

$$(6)\qquad \sum_{\deg F=\nu} \lambda(F) = (-1)^{\nu}$$

where $[\alpha]$ is the greatest integer $\leqq \alpha$.

4. *The divisor functions.* Clearly from the definition of $\tau_\rho(F)$,

$$\sum_F \tau_\rho(F)/|F|^s = (\sum_F 1/|F|^s)^{\rho+1} = (1-p^{\pi}/p^{\pi s})^{-\rho-1}$$

$$= \sum_{\nu=0}^{\infty} \binom{\nu+\rho}{\rho} p^{\pi\nu}/p^{\pi\nu s},$$

whence equating coefficients,

$$(7)\qquad \sum_{\deg F=\nu} \tau_\rho(F) = \binom{\nu+\rho}{\rho} p^{\pi\nu}.$$

The rational analog of this, it will be recalled, is an exceedingly complex asymptotic formula.

Again for $\sigma_\rho(F)$, from

$$\sum_F \sigma_\rho(F)/|F|^s = \sum 1/|F|^s \cdot \sum |F|^\rho/|F|^s$$

$$= \sum p^{\pi\nu}/p^{\pi\nu s} \cdot \sum p^{\pi\nu(\rho+1)}/p^{\pi\nu s}$$

we find without difficulty that

$$(8)\qquad \sum_{\deg F=\nu} \sigma_\rho(F) = p^{\pi\nu}(p^{\pi\rho(\nu+1)}-1)/(p^{\pi\rho}-1).$$

5. *The generalized $\phi$-functions.* In order to evaluate $\phi_\nu(M)$ we start with

$$\sum_{(F,M)=1} 1/|F|^s = \sum_{\nu=0}^{\infty} \phi_\nu(M)/p^{\rho\pi\nu s},$$

where $(F, M)$ is the "greatest" common divisor of $F$ and $M$. But

$$\sum_{(F,M)=1} 1/|F|^s = \prod_{P|M} 1/(1-1/|P|^s) = \zeta(s) \prod_{P|M} (1-1/|P|^s)$$

$$= \sum_{\nu} p^{\pi\nu}/p^{\pi\nu s} \cdot \prod_{P|M} (1-1/|P|^s);$$

therefore

$$\phi_\nu(M) = p^{\pi\nu} \prod_{P|M} (1-1/|P|^s), \tag{9}$$

where in the right member we retain only those terms of the expanded product which are of non-negative degree in $p^\pi$. When $\nu = \deg M$ this coincides with Dedekind's * expression.

For this case it is convenient to rewrite (9) as

$$\phi(M) = \sum_{D|M} \mu\left(\frac{M}{D}\right) |D|;$$

then

$$\sum \phi(M)/|M|^s = \sum \mu(M)/|M|^s \sum |F|/|F|^s$$

$$= (1-p^\pi/p^{\pi s}) \sum_{\nu} p^{2\pi\nu}/p^{\pi\nu s},$$

so that

$$\sum_{\deg F=\nu} \phi(F) = p^{2\pi\nu} - p^{\pi(2\nu-1)}. \tag{10}$$

We pass over the immediate generalization of this to $\phi$-functions of higher order.

6. *Formula for $Q_\rho(\nu)$.*

$$\sum_{\nu} Q_\rho(\nu)/p^{\pi\nu s} = \sum_{P^{\rho+1} \nmid F} 1/|F|^s = \prod_{P} (1 + 1/|P|^s + \cdots + 1/|P|^{\rho s})$$

$$= \prod [1-1/|P|^{(\rho+1)s}]/[1-1/|P|^s] = \zeta(s)/\zeta((\rho+1)s)$$

$$= \sum p^{\pi\nu}/p^{\pi\nu s}(1-p^\pi/p^{\pi s(\rho+1)}),$$

so that

$$\begin{cases} Q_\rho(\nu) = p^{\pi\nu} - p^{\pi(\nu-\rho)} & \text{for } \nu > \rho, \\ Q_\rho(\nu) = p^{\pi\nu} & \text{for } \nu \leqq \rho. \end{cases} \tag{ii}$$

7. *Other functions.* It would be possible to give an indefinite number of additional functions satisfying relations like (5), (7), (8), (10), (11). In general they are more complicated; everything depends of course on the particular combination of $\zeta$'s involved. For example defining $\vartheta(\nu)$ by

$$\sum \vartheta(\nu)/p^{\pi\nu s} = \prod (1 + 1/|P|^{2s} + 1/|P|^{3s} + \cdots)$$

—the arithmetic significance of $\vartheta$ is obvious—we find that

$$\sum \vartheta(\nu)/p^{\pi\nu s} = \zeta(2s)\zeta(3s)/\zeta(6s),$$

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* *Loc. cit.*

and $\vartheta(\nu)$ is found as a rather involved function of $\nu$. On the other hand, defining $\Theta(\nu)$ by

$$\sum \Theta(\nu)/p^{\pi\nu s} = \prod (1 + 1/|P|^{2s} - 1/|P|^{3s} + 1/(P)^{4s} - \cdots)$$
$$= \zeta(2s)/\zeta(3s) = \sum p^{\pi\nu}/p^{2\pi\nu s}(1 - p^{\pi}/p^{3\pi s}),$$

we see that

$$\begin{cases} \Theta(\nu) = (-1)^{\nu} p^{\pi[\nu/2]} & \text{for } \nu \neq 1, \\ \Theta(1) = 0. \end{cases} \tag{12}$$

However $\vartheta$ is the simpler function arithmetically.

It is clear that generally if

$$\sum f(\nu)/p^{\pi\nu s} = \zeta(\alpha s)/[\zeta(\beta s) \cdots \zeta(\gamma s)],$$

we may deduce a simple explicit formula for $f$; and its arithmetic meaning is rather easily grasped. The next case of interest is

$$\sum f(\nu)/p^{\pi\nu s} = [\zeta(\alpha s) \cdots \zeta(\beta s)]/[\zeta(\gamma s) \cdots \zeta(\delta s)],$$

but it is unnecessary to consider this in detail here.

## II. The Reciprocity Theorem.

8. *The Euler Criterion.* Consider the binomial congruence

$$X^{p^{\pi}-1} \equiv A, \bmod P, \quad P \nmid A, \tag{13}$$

$P$ as usual being irreducible and primary. Exactly as in the rational theory we may prove the

*Euler Criterion. A necessary and sufficient condition that* (13) *be solvable in* $\mathfrak{D}$ *is that*

$$\left\{ \frac{A}{P} \right\} = 1,$$

*where* $\{A/P\}$, *the* $(p^{\pi} - 1)$ *-ic power character, is defined by*

$$\left\{ \frac{A}{P} \right\} \equiv A^{(p^{\pi\nu}-1)/(p^{\pi}-1)}, \text{ mod } P \qquad (\deg P = \nu).$$

Evidently

$$\left\{ \frac{AB}{P} \right\} = \left\{ \frac{A}{P} \right\} \left\{ \frac{B}{P} \right\},$$

so that the calculation of $\{A/P\}$ depends upon that of $\{Q/P\}$ and $\{a/P\}$, $Q$ irreducible and primary. For $\{a/P\}$ we shall content ourselves with

$$\left\{\frac{A}{P}\right\} = a^{\nu}, \tag{14}$$

the proof of which is immediate. For $\{Q/P\}$ we have the theorem of reciprocity stated in the Introduction:

$$\left\{\frac{Q}{P}\right\} = (-1)^{\rho\nu}\left\{\frac{Q}{P}\right\}, \qquad \rho = \deg Q. \tag{15}$$

We shall prove this by establishing three lemmas, each of a simple nature.

9. *Analog of Gauss' Lemma.* Let $\mathfrak{R}(A/P)$ denote the remainder in the division of $A$ by $P$. Then we have

LEMMA 1. *If $H$ run through the (primary) polynomials of degree less than the degree of $P$,*

$$\left\{\frac{A}{P}\right\} = \prod_{H} \operatorname{sgn} \mathfrak{R}\left\{\frac{AH}{P}\right\}.$$

Since the proof is so much like the proof of Gauss's Lemma in the ordinary case we shall omit it.

10. LEMMA 2. *Let $A$ be primary of degree $\alpha \geqq \nu$. Then if $P \nmid A$,*

$$\operatorname{sgn} \prod (A - KP) = (-1)^{\alpha-\nu} \operatorname{sgn} \mathfrak{R}\left\{\frac{A}{P}\right\},$$

the product extending over all primary $K$'s of degree $\alpha - \nu$.

Let $K_0$ be the quotient of $A$ by $P$, and put

$$K = K_0 - M, \quad \deg M < \alpha - \nu.$$

Then $A - KP = (A - K_0P) + MP$, so that if $K \neq K$,

$$\deg (A - KP) = \deg MP, \quad \operatorname{sgn} (A - KP) = \operatorname{sgn} MP = \operatorname{sgn} M.$$

We now divide the $M$'s according to degree (excluding $M = 0$). Evidently there are precisely

$$p^{\pi(\gamma+1)} - p^{\pi\gamma}$$

of degree $\gamma$. Now by the extension of Wilson's Theorem the product of the non-zero elements of a $GF(p^{\pi}) = -1$. Hence it is easy to see that

$$\operatorname{sgn} \prod_{\deg M=\rho} M = -1,$$

so that

$$\operatorname{sgn} \prod_{K \neq K_0} (A - KP) = (-1)^{a-\nu};$$

and from this the Lemma follows immediately.

11. LEMMA 3. *If $H$ runs through the primary polynomials of degree $< \nu$ and $K$ those of degree $< \rho$,*

$$\operatorname{sgn} \prod_{H,K} (HQ - KP) = (-1)^{\rho\nu + \operatorname{Min}(\rho^2, \nu^2)} \operatorname{sgn} \prod_H \mathfrak{R}\left\{\frac{HQ}{P}\right\}.$$

Evidently

$$\begin{aligned} \operatorname{sgn} \prod_{H,K} (HQ - KP) &= \operatorname{sgn} \prod_{|HQ|>|KP|} \cdot \operatorname{sgn} \prod_{|HQ|=|KP|} \cdot \operatorname{sgn} \prod_{|HQ|<|KP|} \\ &= \operatorname{sgn} \prod_{|HQ|=|KP|} \cdot \operatorname{sgn} \prod_{|HQ|<|KP|}. \end{aligned} \tag{16}$$

We now consider separately (A). $\rho \geqq \nu$, (B). $\rho < \nu$.

(A). $\rho \geqq \nu$. In order to determine

$$\operatorname{sgn} \prod_{|HQ|=|KP|} (HQ - KP),$$

rewrite this as

$$\prod_H \operatorname{sgn} \prod_K (HQ - KP).$$

Letting $h$, $k$ be the respective degrees of $H$, $K$, and noting that $h + \rho = k + \nu$, we may apply Lemma 2 to the inner product, which accordingly becomes

$$(-1)^{h+\rho-\nu} \operatorname{sgn} \mathfrak{R}\left\{\frac{HQ}{P}\right\}.$$

Then the double product

$$\begin{aligned} &= \prod_H (-1)^{h+\rho-\nu} \cdot \operatorname{sgn} \prod_H \mathfrak{R}\left(\frac{HQ}{P}\right) \\ &= \prod_{h=0}^{\nu-1} (-1)^{(h+\rho-\nu)p^{\nu h}} \cdot \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right) \\ &= (-1)^{\frac{1}{2}\nu(\nu-1)+\rho\nu+p^2} \cdot \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right). \end{aligned} \tag{17}$$

As for

$$\operatorname{sgn} \prod_{|HQ|<|KP|} (HQ - KP),$$

we see from $h + \rho < k + \nu$, that for

$$\begin{aligned} h &= 0, \quad & k &= \rho - \nu + 1, \cdots, \rho - 1, \\ h &= 1, \quad & k &= \rho - \nu + 2, \cdots, \rho - 1, \\ &\cdots\cdots\cdots\cdots\cdots \\ h &= \nu - 2, \quad & k &= \rho - 1. \end{aligned}$$

Therefore

$$\operatorname{sgn} \prod_{|HQ|<|KP|} = \prod_{h=0}^{\nu-2} \Big[ \prod_{k=\rho-\nu+h+1}^{\rho-1} (-1)^{p^{\nu+h}} \Big]^{p^{\nu+h}}$$

$$= (-1)^{\frac{1}{2}\nu(\nu-1)}. \tag{18}$$

Substituting from (17) and (18) in (16),

$$\operatorname{sgn} \prod_{H,K} = (-1)^{\rho\nu+\nu^2} \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right) \qquad (\rho \geqq \nu). \tag{19}$$

(B). $\rho < \nu$.

We now find that

$$\operatorname{sgn} \prod_{|HQ|=|KP|} = \prod_{H} \operatorname{sgn} \prod_{K} (HQ - KP)$$

$$= \prod_{h=\nu-\rho}^{\nu-1} (-1)^{h+\rho-\nu} \cdot \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right)$$

$$= (-1)^{\frac{1}{2}\rho(\rho-1)} \cdot \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right). \tag{20}$$

For the second partial product in (16) we now notice that for

$$\begin{aligned}
&h = 0, && k = 0, \cdots, \rho-1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h = \nu-\rho-1, && k = 0, \cdots, \rho-1,\\
&h = \nu-\rho, && k = 1, \cdots, \rho-1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h = \nu-2, && k = \rho-1;
\end{aligned}$$

accordingly

$$\operatorname{sgn} \prod_{|HQ|<|KP|} = \prod_{h=0}^{\nu-\rho-1} (-1)^{\rho} \cdot \prod_{h=\nu-\rho}^{\nu-2} (-1)^{\nu-h-1}$$

$$= (-1)^{\rho(\nu-\rho)+\frac{1}{2}\rho(\rho-1)}. \tag{21}$$

Substituting from (20) and (21) in (16),

$$\operatorname{sgn} \prod_{H,K} = (-1)^{\rho\nu+\rho^2} \operatorname{sgn} \prod \mathfrak{R}\left(\frac{HQ}{P}\right) \qquad (\rho < \nu). \tag{22}$$

But (19) and (22) imply the truth of Lemma 3.

12. From Lemma 1 and 3 it follows that

$$\left\{ \frac{Q}{P} \right\} = (-1)^{\rho\nu + \operatorname{Min}(\rho^2,\nu^2)} \operatorname{sgn} \prod_{H,K} (HQ - KP).$$

Interchanging $P$ and $Q$,

$$\left\{\frac{P}{Q}\right\} = (-1)^{\rho\nu + \text{Min}\,(\rho^2,\nu^2)} \operatorname{sgn} \prod_{H,K} (KP - HQ).$$

Immediately then

$$\left\{\frac{Q}{P}\right\} = (-1)^{\epsilon} \left\{\frac{P}{Q}\right\},$$

where

$$\epsilon = \frac{p^{\pi\nu}-1}{p^{\pi}-1} \quad \frac{p^{\pi\rho}-1}{p^{\pi}-1} \equiv \rho\nu, \mod 2,$$

thus completing the proof of the theorem of reciprocity.

In the proof of Lemma 2 and 3 we have implicitly supposed $p \neq 2$. However in that case the proofs are even simpler and, since in the $GF(2^{\pi})$, $+1$ and $-1$ are the same, we may here write (15) in the simpler form

$$\left\{\frac{Q}{P}\right\} = \left\{\frac{P}{Q}\right\} \qquad (p = 2). \tag{23}$$

13. *Generalizations.* If $M = P_1 \cdots P_m$, the $P$'s not necessarily being distinct, and $(A, M) = 1$, we define

$$\left\{\frac{A}{M}\right\} = \left\{\frac{A}{P_1}\right\} \cdots \left\{\frac{A}{P_m}\right\}.$$

Let now

$$N = Q_1 \cdots Q_n, \quad \deg P_i = \nu_i, \quad \deg Q_i = \rho_i;$$

then

$$\begin{aligned}\left\{\frac{M}{N}\right\} &= \prod \left\{\frac{Q_i}{P_j}\right\} \\ &= \prod (-1)^{\rho_i\nu_j} \left\{\frac{P_j}{Q_i}\right\} \\ &= (-1)^{(\rho_1+\cdots+\rho_n)(\nu_1+\cdots+\nu_m)} \left\{\frac{M}{N}\right\}.\end{aligned}$$

Putting $\deg N = \rho_1 + \cdots + \rho_m = \rho$, $\deg M = \nu$, we replace (15) by the more general

$$\left\{\frac{M}{N}\right\} = (-1)^{\rho\nu} \left\{\frac{M}{N}\right\}, \tag{24}$$

in which $M$ and $N$ are primary and prime to each other.

If we suppose $\pi = 1$, and define the *quadratic* character

$$\left\{\frac{A}{P}\right\}_2 = \left\{\frac{A}{P}\right\}^{(p-1)/2},$$

then (15) becomes

$$\left\{\frac{Q}{P}\right\}_2 = (-1)^{p\nu(p-1)/2}\left\{\frac{P}{Q}\right\}_2,$$

which is Dedekind's theorem of reciprocity. Similarly we may deduce a theorem of reciprocity of any index $\delta$, a divisor of $p^\pi - 1$.

On the other hand, as remarked in the Introduction, we may also treat the case of index $\delta$, where $\delta \nmid (p^\pi - 1)$. For assuming $p \nmid \delta$, we may find an integer $\pi_1$ that $\pi \mid \pi_1$ and $\delta \mid (p^{\pi_1} - 1)$, and then it is only necessary to extend the domain $\mathfrak{D}(p^\pi, x)$ to the larger domain $\mathfrak{D}(p^{\pi_1}, x)$. The assumption $p \nmid \delta$ is of no moment since

$$A \equiv A^{p^{\pi\nu}} \equiv B^p, \mod P;$$

that is, every polynomial is a $p$-ic residue, mod $P$.

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# DIVISION ALGEBRAS ASSOCIATED WITH AN EQUATION WHOSE GROUP HAS FOUR GENERATORS.

By MINA S. REES.

1. *Introduction.* The construction of all division algebras is the outstanding problem in the theory of linear algebras. In the history of this problem, the procedure has been to examine first the necessary and sufficient conditions that the constructed algebra be associative; and second, the conditions that it be a division algebra. Since L. E. Dickson's announcement [1] in 1905 of his discovery of a system of non-commutative division algebras, he has published three papers examining associativity conditions for such algebras. In the first of these papers,[2] algebras associated with cyclic equations were discussed; in the second,[3] general results were obtained for the abelian non-cyclic case, and for that type of non-abelian case where the Galois group of the basic equation was a solvable group. This study was carried forward by Williamson,[4] in a paper in which detailed associativity conditions were established for the general case of a two generator group, and for a significant special case of the three generator group. In a third paper in 1930, Dickson [5] gave a simplification of his previous method by which he reached specific results for the general case of the two and three generator problems. The second line of investigation, the study of the conditions under which a given associative algebra is a division algebra, has been advanced by Dickson,[6] Wedderburn,[7] and Albert.[8] In addition to these two aspects of the problem, there is a third phase which is concerned with normal division algebras. It is known that any division algebra can be normalized by a suitable

1. *Bulletin of the American Mathematical Society*, Vol. 22 (1905-06), p. 442.

2. "Linear Associative Algebras and Abelian Equations," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 31-46.

3. "New Division Algebras," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 207-34.

4. "Associativity of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 111-25.

5. "Construction of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 319-34.

6. See notes (2) and (3).

7. "A Type of Primitive Algebra," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 162-166.

8. Abstracts in the *Bulletin of the American Mathematical Society*, Vol. 36 (1930), p. 45; p. 198; p. 804.

extension of the field, so that its order is the square of an integer,[9] and a third line of study has established that all division algebras of orders four,[10] nine,[11] or sixteen[12] are among those for which associativity conditions have been found.

The present paper continues the investigation of associativity conditions, by considering the algebras related to an equation whose Galois group has four independent generators. Using Dickson's most recent simplification of his previous method,[13] we set up necessary and sufficient conditions that these algebras be associative. Since the method is inductive, all the results which Dickson obtained for the two or three generator cases[14] will be presupposed. The notation of the present paper is the same as in Dickson's except that $v, y, z, w, \epsilon, \delta$ of his paper are replaced by $z_1, w_1, z_2, w_2, \delta_1, \delta_2$ respectively. Numbered theorems and formulas in square brackets refer to theorems and formulas in his paper.

2. *Algebra $\Sigma$ derived for an equation whose group has four generators.* Let $f(\xi)=0$ be an equation of degree $pqsh$ which is irreducible in $F$ and has the roots

$$(1) \qquad \chi^{a_1}\{\psi^{a_2}[\phi^{a_3}(\theta^{a_4}(i))]\} \qquad (a_1 < h,\ a_2 < s,\ a_3 < p,\ a_4 < q),$$

where $\chi, \psi, \phi, \theta$ are polynomials in $i$ with coefficients in $F$, and the superscripts denote iteratives, not exponents.

Throughout this paper, the brackets { } will mean that each of the included symbols is to be interpreted as a polynomial in the succeeding symbols, where the final argument $i$ is suppressed. Thus $\psi\{\theta\phi\}$ means $\psi[\theta(\phi(i))]$.

Let the roots satisfy the following relations:

$$(2) \qquad \theta^q = i, \quad \phi^p = \theta^e, \quad \psi^s = \phi^b\{\theta^a\}, \quad \chi^h = \psi^l\{\phi^d\theta^c\},$$

$$(3) \qquad \theta\{\phi\} = \phi\{\theta^x\}, \quad \theta\{\psi\} = \psi\{\phi^{z_1}\theta^{w_1}\}, \quad \phi\{\psi\} = \psi\{\phi^{z_2}\theta^{w_2}\},$$

$$(4) \qquad \theta\{\chi\} = \chi\{\psi^{u_1}\phi^{v_1}\theta^{y_1}\}, \quad \phi\{\chi\} = \chi\{\psi^{u_2}\phi^{v_2}\theta^{y_2}\}, \quad \psi\{\chi\} = \chi\{\psi^{u_3}\phi^{v_3}\theta^{y_3}\},$$

where $q, p, s, h$ are the least positive integers for which relations of the type (2) hold.

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9. Dickson, *Algebren und ihre Zahlentheorie*, p. 138.

10. See note (9) above. *Loc. cit.*, p. 46.

11. Wedderburn, "On Division Algebras," *Transactions of the American Mathematical Society*, Vol. 22 (1921), pp. 129-35.

12. Albert, "A Determination of All Division Algebras in Sixteen Units," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 253-60.

13. See note (5), page 51.

14. *Ibid.*

Let $\mathfrak{A}$ be the algebra over $F$ having the basis $i^m j^n k^t o^f$ $(m \leqq pqsh-1,$ $n \leqq q-1,\ t \leqq p-1,\ f \leqq s-1)$ such that $f(i)=0$ and

$$(5) \qquad j^q = g(i), \qquad j^r P(i) = P(\theta^r) j^r;$$

$$(6) \qquad k^p = \beta(i) j^e, \quad k^r P(i) = P(\phi^r) k^r, \quad k^r j^t = [j^{(r)}]^t k^r,$$

where $j^{(r)}$ is defined by [16];

$$(7) \qquad \begin{aligned} o^s = \sigma(i) j^a k^b, \quad & o^r P(i) = P(\psi^r) o^r, \\ & o^r j^t = [j^{(r)}]^t o^r, \\ & o^r k^t = [k^{(r)}]^t o^r, \end{aligned}$$

where

$$J \equiv j' \equiv \delta_1 j^{w_1} k^{z_1}, \qquad K \equiv k' \equiv \delta_2 j^{w_2} k^{z_2}$$

and

$$j^{(r)} = \delta_1 (\psi^{r-1}) (J^{w_1} K^{z_1})^{(r-2)}, \qquad k^{(r)} = \delta_2 (\psi^{r-1}) (J^{w_2} K^{z_2})^{(r-2)}$$

are found by [64]. We assume $g$, $\beta$, and $\sigma$ to be different from zero.

$\mathfrak{A}$ may be obtained from [Theorem 5] by replacing $F$ by the field $F_1$, derived by adjoining to $F$ the elementary symmetric functions of all of the roots $\psi^{a_2}\{\phi^{a_3}\theta^{a_4}\}$. Then the latter are roots of an equation irreducible in $F_1$ of degree $pqs$.[15] Therefore by [Theorem 5], $\mathfrak{A}$ has the above basis. Assume the conditions required in [Theorem 5] so that $\mathfrak{A}$ be associative. We see that (5), (6), (7) are consequences of the associative law and the following set $R$ of relations:

$$(8) \qquad j^q = g(i), \qquad ji = \theta(i) j, \qquad f(i) = 0,$$

$$(9) \qquad k^p = \beta(i) j^e, \qquad ki = \phi(i) k, \qquad kj = \alpha(i) j^x k,$$

$$(10) \qquad o^s = \sigma(i) j^a k^b, \quad oi = \psi(i) o, \quad oj = \delta_1(i) j^{w_1} k^{z_1} o, \quad ok = \delta_2(i) j^{w_2} k^{z_2} o.$$

These relations enable us to reduce any product $AB$ to $C$ of the same form, where $A$, $B$, and $C$ are polynomials of $\mathfrak{A}$. Let $R'$ denote the set of like relations, with $i$, $j$, $k$, $o$ replaced by $i'$, $j'$, $k'$, $o'$ which are defined to be

$$(11) \qquad \begin{aligned} I = i' = \chi(i), \qquad & J = j' = \epsilon_1(i) j^{v_1} k^{v_1} o^{u_1}, \\ & K = k' = \epsilon_2(i) j^{v_2} k^{v_2} o^{u_2}, \\ & O = o' = \epsilon_3(i) j^{v_3} k^{v_3} o^{u_3}. \end{aligned}$$

If relations $R'$ hold, it follows[16] that

$$C' = (AB)' = A'B', \qquad (A+B)' = A' + B'.$$

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15. See note (3), page 51. *Loc. cit.*, § 6.

16. For further details see Dickson, "Construction of Division Algebras," *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 322.

Thus [Theorem 1] will enable us to construct an associative algebra $\Gamma$ provided the correspondence defined in $\mathfrak{S}$ is such that the following conditions are satisfied:

(a) Relations $R'$ hold, so that

$$(AB)^{(r)} = A^{(r)}B^{(r)}, \qquad (A+B)^{(r)} = A^{(r)} + B^{(r)}.$$

(b) An element $\gamma$ is defined for which $\gamma = \gamma'$.

(c) $A^{(h)}\gamma = \gamma A$ for every $A$ in $\mathfrak{S}$.

3. *Associativity Conditions for* $\Gamma$. We investigate first the conditions for $R'$ to be true. It is easily seen that $(8'_2)$, $(9'_2)$ and $(10'_2)$ are satisfied identically by reason of the definition of the correspondence in $\mathfrak{S}$. Insert the values (11) in $(8'_2)$. Then, by $(4_1)$,

$$\begin{aligned} JI &= \epsilon_1\chi\,\{\psi^{u_1}\phi^{v_1}\theta^{y_1}\}\, j^{y_1}k^{v_1}o^{u_1}, \\ &= \epsilon_1\theta\{\chi\}j^{y_1}k^{v_1}o^{u_1} = \theta\{\chi\}J. \end{aligned}$$

Similarly, by using subscripts 2 and 3, relations $(9'_2)$ and $(10'_2)$ are established by $(4_2)$ and $(4_3)$ respectively.

*Preliminary Definitions and Formulas.* We shall use the following abbreviations, corresponding to [36], [38], [39] respectively:

$$B_{m,n}(i) = \pi_n(\phi^{m-1})\pi_{x^n}(\phi^{m-2})\pi_{x^{2n}}(\phi^{m-3})\cdots\pi_{x^{m-1}n}(i), \tag{12}$$

$$\pi_n(i) = \alpha\alpha(\theta^x)\cdots\alpha(\theta^{(n-1)x}), \qquad B_{m,0} = B_{0,n} = 1,$$

$$h(t) \equiv \sum_{n=0}^{t-1} x^{nz_1}w_1, \qquad h(0) = 0, \tag{13}$$

$$C_d(i) \equiv \prod_{n=0}^{d-1} \delta_1\{\phi^{nz_1}\theta^{h(n)}\}B_{nz_1,w_1}\{\theta^{h(n)}\}, \qquad C_0(i) \equiv 1, \tag{14}$$

and the formulas [35] and [37]

$$k^m j^n = B_{m,n}j^{x^m n}k^m, \tag{15}$$

$$\delta j^w k^z \epsilon j^y k^v = \delta\epsilon\{\phi^z\theta^w\}B_{z,y}(\theta^w)\, j^{w+x^z y}k^{z+v}, \tag{16}$$

where $\delta$ and $\epsilon$ are polynomials in $i$. Then from $(10_3)$ we see by induction on $n$, that

$$oj^n = C_n(i)\, j^{h(n)}k^{nz_1}o. \tag{17}$$

Similarly, if $H(n)$ is derived from $h(n)$ by using subscripts 2 throughout, and $D_d(i)$ from $C_d(i)$ by using subscripts 2 and substituting $H(n)$ for $h(n)$, we have

(18) $$ok^n = D_n(i)\, j^{H(n)} k^{nz_2} 0.$$

Formula [58] is

$$t(m,n) \equiv h(m) + x^{mz_1} H(n).$$

From (17) and (18), by induction on $m$, we can deduce formulas corresponding to (15). We find, in fact

(20) $$\begin{cases} o^m j^n = \mathcal{A}_{mn}(i)\, j^{A(m,n)} k^{L(m,n)} o^m, \\ o^m k^n = \mathcal{D}_{mn}(i)\, j^{R(m,n)} k^{S(m,n)} o^m, \end{cases}$$

where (21)-(24) hold. The recursion formulas

(21) $$\begin{cases} A(1,n) = h(n), & A(m+1,n) = t\,[A(m,n),\ L(m,n)], \\ L(1,n) = nz_1, & L(m+1,n) = A(m,n)z_1 + L(m,n)z_2, \end{cases}$$

(22) $$\begin{cases} R(1,n) = H(n), & R(m+1,n) = t\,[R(m,n),\ S(m,n)], \\ S(1,n) = nz_2, & S(m+1,n) = R(m,n)z_1 + S(m,n)z_2, \end{cases}$$

with

$$A(0,n) = S(0,n) = n, \qquad L(0,n) = R(0,n) = 0,$$
$$A(m,0) = L(m,0) = R(m,0) = S(m,0) = 0;$$

and the abbreviations

(23) $$F_{m,n}(i) = C_m D_n \{\phi^{mz_1} \theta^{h(m)}\} B_{mz_1, H(n)} \{\theta^{h(m)}\},$$

(24) $$\begin{cases} \mathcal{A}_{mn}(i) = \prod_{r=0}^{m-1} F_{A(r,n),\ L(r,n)} \{\psi^{m-1-r}\}, \\ \mathcal{D}_{mn}(i) = \prod_{r=0}^{m-1} F_{R(r,n),\ S(r,n)} \{\psi^{m-1-r}\}, \end{cases}$$

with

$$\mathcal{A}_{1n}(i) = C_n(i), \quad \mathcal{D}_{1n}(i) = D_n(i), \quad \mathcal{A}_{0n}(i) \equiv \mathcal{D}_{0n}(i) \equiv 1,$$

actually define the functions that occur in (20).

Applying $(7_2)$, (20), (16), we have

$$\begin{aligned} KJ &= \epsilon_2 j^{y_2} k^{v_2} o^{u_2} \epsilon_1 j^{y_1} k^{v_1} o^{u_1}, \\ &= \epsilon_2 \epsilon_1 \{\psi^{u_2} \phi^{v_2} \theta^{y_2}\}\, \mathcal{A}_{u_2 y_1} \{\phi^{v_2} \theta^{y_2}\}\, \mathcal{D}_{u_2 v_1} \{\phi^{v_2 + L(u_2,y_1)} \theta^{y_2 + x^{v_2} A(u_2,y_1)}\} \\ &\quad \times B_{v_2, A(u_2,y_1)} \{\theta^{y_2}\} B_{v_2 + L(u_2,y_1),\ R(u_2,v_1)} \{\theta^{y_2 + x^{v_2} A(u_2,y_1)}\} \\ &\quad \times j^{y_2 + x^{v_2} A(u_2,y_1) + x^{v_2 + L(u_2,y_1)} R(u_2,v_1)} k^{v_2 + L(u_2,y_1) + S(u_2,v_1)} o^{u_2 + u_1}. \end{aligned} \tag{25$_1$}$$

Define

$$\mathcal{K}_{ab}(r,t) \equiv v_a + \sum_{n=1}^{r-1} L(nu_a, y_b) + \sum_{m=1}^{t-1} S(mu_a, v_b), \qquad \mathcal{K}_{ab}(0,0) = 0,$$

(26)

$$\begin{aligned} \mathcal{Q}_{ab}(r,t) \equiv y_a &+ \sum_{n=1}^{r-1} x^{\mathcal{K}_{ab}(n,n)} A(nu_a, y_b) \\ &+ \sum_{m=1}^{t-1} x^{\mathcal{K}_{ab}(m+1,m)} R(mu_a, v_b), \qquad \mathcal{Q}_{ab}(0,0) = 0, \end{aligned}$$

where summations from one to zero are equal to zero;

$$(27) \qquad \mathcal{K}_{ab}(2,2) \equiv \mathcal{F}_{ab}, \qquad \mathcal{Q}_{ab}(2,2) \equiv {}_{ab},$$

$$(28) \quad \mathcal{H}_{ab}(i) \equiv \epsilon_a\epsilon_b\{\psi^{u_a}\phi^{v_a}\theta^{y_a}\}\, A_{u_av_b}\{\phi^{v_a}\theta^{y_a}\}$$
$$\times \mathcal{D}_{u_av_b}\{\phi^{\mathcal{K}^{ab}(2,1)}\theta^{\mathcal{Q}_{ab}(2,1)}\}B_{v_a,A(u_a,y_b)}\{\theta^{y_a}\}$$
$$\times B_{\mathcal{K}ab(2,1),\,R(u_a,v_b)}\{\theta^{\mathcal{Q}_{ab}(2,1)}\};$$

$$(29) \quad \mathcal{G}^{(d)}(i) \equiv \prod_{n=0}^{d-1} \epsilon\{\psi^{nu}\phi^{\mathcal{K}(n,n)\theta^{\mathcal{Q}(n,n)}}\}\, A_{nu,y}\{\phi^{\mathcal{K}(n,n)\theta^{(n,n)}}\}$$
$$\times B_{\mathcal{K}(n,n),\,A(nu,y)}\{\theta^{\mathcal{Q}(n,n)}\}B_{\mathcal{K}(n+1,n),\,R(nu,v)}\{\theta^{\mathcal{Q}(n+1,n)}\}$$
$$\times \mathcal{D}_{nu,v}\{\phi^{\mathcal{K}(n+1,n)\theta^{\mathcal{Q}(n+1,n)}}\} \qquad (d>0),$$

$$\mathcal{G}^{(0)}(i) \equiv 1, \qquad \mathcal{G}^{(1)}(i) = \epsilon,$$

where a subscript on the $\mathcal{G}$ is to be interpreted as the subscript of every $\epsilon$, $u$, $v$, $y$, and the double subscript of every $\mathcal{K}$, $\mathcal{Q}$. Using these abbreviations we obtain a generalized statement of $(25_1)$

$$(25_2) \qquad \epsilon_a j^{y_a}k^{v_a}{}_O{}^{u_a}\epsilon_b j^{y_b}k^{v_b}{}_O{}^{u_b} = \mathcal{H}_{ab}(i)\, j^{\mathcal{E}_{ab}}k^{\mathcal{F}_{ab}}{}_O{}^{u_a+u_b}.$$

We have immediately

$$(30) \qquad \begin{aligned} KJ &= \mathcal{H}_{21}(i)\, j^{\mathcal{E}_{21}}k^{\mathcal{F}_{21}}{}_O{}^{u_2+u_1}, \\ OJ &= \mathcal{H}_{31}(i)\, j^{\mathcal{E}_{31}}k^{\mathcal{F}_{31}}{}_O{}^{u_3+u_1}, \\ OK &= \mathcal{H}_{32}(i)\, j^{\mathcal{E}_{32}}k^{\mathcal{F}_{32}}{}_O{}^{u_3+u_2}. \end{aligned}$$

By induction on $d$, we obtain from $(11_2)$ and $(25_1)$

$$(31) \qquad J^d = \mathcal{G}_1{}^{(d)}\, j^{\mathcal{Q}_{11}(d,d)}k^{\mathcal{K}_{11}(d,d)}{}_O{}^{u_1d}.$$

$K^d$ and $O^d$ are obtained from $J^d$ by using subscripts 2 and 3 respectively.

In the discussion which follows, the numbers in the left-hand column of the table given below will be used as superscripts on $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$, to indicate the results obtained from formulas (27) and (28) by making the tabulated substitutions. In (7) and (8), subscripts $(n,t)$ and $(m,n,t)$ respectively, will be used on $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$. The last line of the table gives superscripts (9), (10), (11) when subscripts 1, 2, 3 respectively are used on $y$, $v$, and $u$.

| | Superscript | $\epsilon_a$ | $y_a$ | $v_a$ | $u_a$ | $\epsilon_b$ | $y_b$ | $v_b$ | $u_b$ |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | $\mathcal{G}_1^{(x)}$ | $\mathcal{Q}_{11}(x, x)$ | $\mathcal{K}_{11}(x, x)$ | $u_1 x$ | $\epsilon_2$ | $y_2$ | $v_2$ | $u_2$ |
| | (2) | $\mathcal{G}_2^{(z_1)}$ | $\mathcal{Q}_{22}(z_1 z_1)$ | $\mathcal{K}_{22}(z_1, z_1)$ | $u_2 z_1$ | $\epsilon_3$ | $y_3$ | $v_3$ | $u_3$ |
| | (3) | $\mathcal{G}_1^{(w_1)}$ | $\mathcal{Q}_{11}(w_1, w_1)$ | $\mathcal{K}_{11}(w_1, w_1)$ | $u_1 w_1$ | $\mathcal{H}^{(2)}$ | $\mathcal{E}^{(2)}$ | $\mathcal{F}^{(2)}$ | $u_2 z_1 + u_3$ |
| | (4) | $\mathcal{G}_2^{(z_2)}$ | $\mathcal{Q}_{22}(z_2, z_2)$ | $\mathcal{K}_{22}(z_2, z_2)$ | $u_2 z_2$ | $\epsilon_3$ | $y_3$ | $v_3$ | $u_3$ |
| (32) | (5) | $\mathcal{G}_1^{(w_2)}$ | $\mathcal{Q}_{11}(w_2, w_2)$ | $\mathcal{K}_{11}(w_2, w_2)$ | $u_1 w_2$ | $\mathcal{H}^{(4)}$ | $\mathcal{E}^{(4)}$ | $\mathcal{F}^{(4)}$ | $u_2 z_2 + u_3$ |
| | (6) | $\mathcal{G}_1^{(a)}$ | $\mathcal{Q}_{11}(a, a)$ | $\mathcal{K}_{11}(a, a)$ | $u_1 a$ | $\mathcal{G}_2^{(b)}$ | $\mathcal{Q}_{22}(b, b)$ | $\mathcal{K}_{22}(b, b)$ | $u_2 b$ |
| | (7) Subscript $(n, t)$ | $\mathcal{G}_2^{(n)}$ | $\mathcal{Q}_{22}(n, n)$ | $\mathcal{K}_{22}(n, n)$ | $u_2 n$ | $\mathcal{G}_3^{(t)}$ | $\mathcal{Q}_{33}(t, t)$ | $\mathcal{K}_{33}(t, t)$ | $u_3 t$ |
| | (8) Subscript $(m, n, t)$ | $\mathcal{G}_1^{(m)}$ | $\mathcal{Q}_{11}(m, m)$ | $\mathcal{K}_{11}(m, m)$ | $u_1 m$ | $\mathcal{H}^{(7)}{}_{(n,t)}$ | $\mathcal{E}^{(7)}{}_{(n,t)}$ | $\mathcal{F}^{(7)}{}_{(n,t)}$ | $u_2 n + u_3 t$ |
| | (9), (10), or (11) | $\Omega_{y,v,u,h-2}$ | $\eta$ | $\lambda$ | $\tau$ | $\rho$ | $c$ | $d$ | $l$ |

where the arguments of $\eta$, $\lambda$, $\tau$, in (9), (10), (11) are $(y, v, u, h-2)$. We shall need the formula

$$k^{np} = \beta_n(i) j^{en}, \qquad \beta_n(i) = \beta\beta(\theta^e) \cdots \beta(\theta^{e(n-1)}), \tag{33}$$

which is [42] and the formula

$$o^{ns} = \sigma_n(i) j^{r(n)} k^{bn}, \tag{34}$$

where

$$r(n) = \sum_{t=0}^{n-1} x^{tb} a, \qquad r(0) = 0,$$

$$\sigma_n(i) = \prod_{t=0}^{n-1} \sigma\{\phi^{tb} \theta^{r(t)}\} B_{tb,a}\{\theta^{r(t)}\}, \qquad \sigma_0(i) \equiv 1.$$

which is obtained by induction on $s$, using $(7_1)$ and (16).

In finding the associativity conditions for our algebra, we must discuss ten conditions of the type

$$P(i) j^U k^V o^W = S(i) j^X k^Y o^Z. \tag{35}$$

Since $s$ is the least power of $o$ which gives a polynomial in $i$, $j$, $k$, we have

$$Z - W = c_1 s, \qquad c_1 \text{ an integer.} \tag{36}$$

Replacing $o^{c_1 s}$ by its value (34), and applying (16), we have

$$\begin{aligned} P(i) j^U k^V &= S(i)\, j^X k^Y \sigma_{c_1}(i) j^{r(c_1)} k^{bc_1}, \\ &= S(i) \sigma_{c_1}\{\phi^Y \theta^X\} B_{Y, r(c_1)}\{\theta^X\} j^{X + xYr(c_1)} k^{Y + bc_1}. \end{aligned}$$

By comparing exponents of $k$, since $p$ is the least power of $k$ which gives a polynomial in $i$ and $j$, we see that

$$Y + bc_1 - V = c_2 p, \qquad c_2 \text{ an integer.} \tag{37}$$

Replacing $k^{c_2 p}$ by its value (33), applying $(5_2)$, and comparing exponents of $j$, we have

$$X + x^Y r(c_1) + ec_2 - U = c_3 q, \qquad c_3 \text{ an integer.} \tag{38}$$

Replacing $j^{c_3 q}$ by its value from $(5_1)$, we see that (35) is equivalent to the associativity condition

$$P(i) = S(i) \sigma_{c_1}\{\phi^Y \theta^X\} B_{Y, r(c_1)}\{\theta^X\} \beta_{c_2}\{\theta^{X + xYr(c_1)}\} g^{c_3}. \tag{39}$$

In applying these results to special cases, identify the polynomials $P(i)$ and $S(i)$ with the given coefficients, and the exponents $U$, $V$, $W$, $X$, $Y$, $Z$, with

the given exponents in such a way that the $c_1$ determined by (36) is positive. Then it is formally possible that $c_2$ determined by (37) is negative. In this case we would have

$$V - Y - bc_1 = -c_2 p.$$

Proceeding as above, replacing $k^{-c_2 p}$ by its value (33) and comparing exponents of $k$, we find

$$X + x^Y r(c_1) - U + ec_2 = c_3 q$$

which agrees with (38). The associativity condition for this case is

$$P(i)\beta_{-c_2}\{\theta^U\} = S(i)\sigma_{c_1}\{\phi^Y\theta^X\}B_{Y,r(c_1)}\{\theta^X\}g^{c_3}. \tag{40}$$

We observe, therefore, that if we define

$$\beta_{-n} \equiv [\beta_n\{\theta^{-en}\}]^{-1} \tag{41}$$

relation (40) follows from (39) since

$$X + x^Y r(c_1) + ec_2 \equiv U \pmod q, \qquad \theta^q = i.$$

We may now apply (36), (37), (38), (39), (41) as formulas to determine the desired associativity conditions. The relations corresponding to (35) will be stated, and the evaluation of the formulas left for special cases. Thus the associativity conditions for our algebra will be derivable from relations (42), (43), (44), (45), (46), (47), (50), (59), (60), (61) given below, by the application of formulas (39), (41).

We will discuss first $(9_3')$, $(10_3')$, $(10_4')$, $(8_1')$, $(9_1')$, $(10_1')$.

*Condition that* $KJ = \alpha(\chi)J^x K$. By $(30_1)$, (31) and (32), this reduces to

$$\mathfrak{H}_{21}(i)j^{\mathfrak{E}_{21}}k^{\mathfrak{F}_{21}}{}_0{}^{u_2+u_1} = \alpha(\chi)\mathfrak{G}_1{}^{(x)}(i)j^{\mathfrak{L}_{11}(x,x)}k^{\mathfrak{K}_{11}(x,x)}{}_0{}^{u_1 x}\epsilon_2(i)j^{v_3}k^{v_2}{}_0{}^{u_2}.$$

Hence

$$\mathfrak{H}_{21}(i)j^{\mathfrak{E}_{21}}k^{\mathfrak{F}_{21}}{}_0{}^{u_2+u_1} = \alpha(\chi)\mathfrak{H}^{(1)}(i)j^{\mathfrak{E}^{(1)}}k^{\mathfrak{F}^{(1)}}{}_0{}^{u_1 x+u_2}. \tag{42}$$

*Condition that* $OJ = \epsilon_1(\chi)J^{w_1}K^{z_1}O$ *(subscripts* 1 *on* $w$ *and* $z$*).*

$$\mathfrak{H}_{31}(i)j^{\mathfrak{E}_{31}}k^{\mathfrak{F}_{31}}{}_0{}^{u_3+u_1} = \epsilon_1(\chi)\mathfrak{G}_1{}^{(w)}j^{\mathfrak{L}_{11}(w,w)}k^{\mathfrak{K}_{11}(w,w)}{}_0{}^{u_1 w} \times \mathfrak{G}_2{}^{(z)}j^{\mathfrak{L}^{(zz)}}k^{\mathfrak{K}_{21}(z,z)}{}_0{}^{u_2 z}\epsilon_3 j^{v_3}k^{v_3}{}_0{}^{u_3}.$$

Hence

$$\mathfrak{H}_{31}(i)j^{\mathfrak{E}_{31}}k^{\mathfrak{F}_{31}}{}_0{}^{u_3+u_1} = \epsilon_1(\chi)\mathfrak{H}^{(3)}j^{\mathfrak{E}^{(3)}}k^{\mathfrak{F}^{(3)}}{}_0{}^{u_1 w_1+u_2 z_1+u_3}. \tag{43}$$

*Condition that* $OK = \epsilon_2(\chi)J^{w_2}K^{z_2}O$ *(subscripts* 2 *on* $w$ *and* $z$*).*

$$(44)\qquad \mathcal{H}_{32}(i)j^{\mathcal{E}_{32}}k^{\mathcal{F}_{32}}o^{u_3+u_2} = \epsilon_2(\chi)\,\mathcal{H}^{(5)}j^{\mathcal{E}^{(5)}}k^{\mathcal{F}^{(5)}}o^{u_1w_2+u_3z_2+u_3}.$$

*Condition that* $J^q = g(\chi)$.

$$(45)\qquad \mathcal{G}_1{}^{(q)}j^{\mathcal{Q}_{11}(q,q)}k^{\mathcal{K}_{11}(q,q)}o^{u_1q} = g(\chi).$$

*Condition that* $K^p = \beta(\chi)J^e$.

$$(46)\qquad \mathcal{G}_2{}^{(p)}j^{\mathcal{Q}_{22}(p,p)}k^{\mathcal{K}_{22}(p,p)}o^{u_2p} = \beta(\chi)\,\mathcal{G}_1{}^{(e)}j^{\mathcal{Q}_{11}(e,e)}k^{\mathcal{K}_{11}(e,e)}o^{u_1e}.$$

*Condition that* $O^s = \sigma(\chi)J^aK^b$.

$$\mathcal{G}_3{}^{(s)}j^{\mathcal{Q}_{33}(s,s)}k^{\mathcal{K}_{33}(s,s)}o^{u_3s} = \sigma(\chi)\mathcal{G}_1{}^{(a)}j^{\mathcal{Q}_{11}(a,a)}k^{\mathcal{K}_{11}(a,a)}o^{u_1a}\,\mathcal{G}_2{}^{(b)}j^{\mathcal{Q}_{22}(b,b)}k^{\mathcal{K}_{22}(b,b)}o^{u_2b}.$$

Hence

$$(47)\qquad \mathcal{G}_3{}^{(s)}j^{\mathcal{Q}_{33}(s,s)}k^{\mathcal{K}_{33}(s,s)}o^{u_3s} = \sigma(\chi)\,\mathcal{H}^{(6)}j^{\mathcal{E}^{(6)}}k^{\mathcal{F}^{(6)}}o^{u_1a+u_2b}.$$

We have now given all the conditions derived from (8′), (9′), (10′), in the form (35). Choose as $\gamma$,

$$(48)\qquad \gamma = \rho(i)j^ck^do^l, \qquad [\rho(i) \neq o].$$

Using (25), (31), (32), we establish the formula

$$(49)\qquad J^mK^nO^t = \mathcal{H}^{(8)}_{(m,n,t)}j^{\mathcal{E}^{(8)}_{(m,n,t)}}k^{\mathcal{F}^{(8)}_{(m,n,t)}}o^{u_1m+u_2n+u_3t}.$$

*Condition that* $\gamma = \gamma'$ [*subscripts* (*c, d, l*) *on* $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$].

$$\rho(i)j^ck^do^l = \rho(\chi)J^cK^dO^l,$$

$$(50)\qquad \rho(i)j^ck^do^l = \rho(\chi)\mathcal{H}^{(8)}j^{\mathcal{E}^{(8)}}k^{\mathcal{F}^{(8)}}o^{u_1c+u_2d+u_3l}.$$

In order to construct our associative algebra $\Gamma$, we need the further condition that

$$(51)\qquad A^{(h)}\gamma = \gamma A$$

for all $A$ in $\Sigma$. Evidently this condition holds for $A = P(i)$, by reason of our definition of $\gamma$. For, by (5), (6), (7), and ($2_4$),

$$\gamma P(i) = \rho P\{\psi^l\phi^d\theta^c\}j^ck^do^l,$$
$$= P(\chi^h)\gamma = P^{(h)}\gamma.$$

We must now find the condition that (51) holds when $A$ takes the values $j$, $k$, $o$, respectively.

*Condition that* $j^{(h)}\gamma = \gamma j$ (*subscripts* 1 *on every* $\epsilon$, $u$, $v$, $y$).

$$j' = \epsilon j^yk^vo^u, \qquad j'' = \epsilon(\chi)J^yK^vO^u.$$

We need a formula of the type

$$(52)\qquad (J^m K^n O^t)^{(r)} = \Omega_{(m,n,t,r)}(i)\, j^{\eta(m,n,t,r)}\, k^{\lambda(m,n,t,r)}\, o^{\tau(m,n,t,r)}.$$

Hence

$$(53)\qquad \Omega_{(m,n,t,r)}(\chi) J^{\eta(m,n,t,r)} K^{\lambda(m,n,t,r)} O^{\tau(m,n,t,r)}$$

must be identically equal to

$$(54)\qquad \Omega_{(m,n,t,r+1)}(i)\, j^{\eta(m,n,t,r+1)}\, k^{\lambda(m,n,t,r+1)}\, o^{\tau(m,n,t,r+1)}.$$

But by (49), (53) reduces to

$$\Omega_{(m,n,t,r)}(\chi)\, \mathcal{H}^{(8)}_{(\eta,\lambda,\tau)}(i)\, j^{\mathcal{E}^{(8)}_{(\eta,\lambda,\tau)}}\, k^{\mathcal{F}^{(8)}_{(\eta,\lambda,\tau)}}\, o^{u_1\eta+u_2\lambda+u_3\tau},$$

where the arguments of $\eta$, $\lambda$, $\tau$ are $(m, n, t, r)$. Therefore we have the recursion formulas

$$(55)\qquad \begin{aligned} \tau(m,n,t,r+1) &= u_1\eta + u_2\lambda + u_3\tau,\\ \lambda(m,n,t,r+1) &= \mathcal{F}^{(8)}_{(\eta,\lambda,\tau)},\\ \eta(m,n,t,r+1) &= \mathcal{E}^{(8)}_{(\eta,\lambda,\tau)},\\ \Omega_{(m,n,t,r+1)}(i) &= \Omega_{(m,n,t,r)}(\chi)\, \mathcal{H}^{(8)}_{(\eta,\lambda,\tau)}, \end{aligned}$$

where $\eta$, $\lambda$, $\tau$ have arguments $(m, n, t, r)$. Since we may identify (71) for $r = 0$ with (60), we have the initial values

$$(56)\qquad \begin{aligned} \tau(m,n,t,o) &= u_1 m + u_2 n + u_3 t,\\ \lambda(m,n,t,o) &= \mathcal{F}^{(8)}_{(m,n,t)},\\ \eta(m,n,t,o) &= \mathcal{E}^{(8)}_{(m,n,t)},\\ \Omega_{(m,n,t,o)}(i) &= \mathcal{H}^{(8)}_{(m,n,t)}(i). \end{aligned}$$

For these initial values, (55) and (56) serve as formulas to determine $\tau$, $\lambda$, $\eta$, $\Omega$ completely. Now

$$(57)\qquad \begin{aligned} j^{(h)}\gamma &= [\epsilon(\chi) J^y K^v O^u]^{(h-2)} \rho(i)\, j^c k^d o^l,\\ &= \epsilon(\chi^{h-1})\, \mathcal{H}^{(9)} j^{\mathcal{E}^{(9)}} k^{\mathcal{F}^{(9)}} o^{\tau(y,v,u,h-2)+l}. \end{aligned}$$

where every $\epsilon$, $y$, $v$, $u$, has subscript 1. By $24_1$

$$(58)\qquad \begin{aligned} \gamma j &= \rho j^c k^d o^l j\\ &= \rho \mathcal{A}_{l1}\{\phi^d \theta^c\} B_{d,A(l,1)}\{\theta^c\} j^{c+x^d A(l,1)} k^{d+L(l,1)} o^l. \end{aligned}$$

The condition that $j^{(h)}\gamma = \gamma j$ is therefore equivalent to

$$(59)\qquad \begin{aligned} &\epsilon_1(\chi^{h-1})\, \mathcal{H}^{(9)} j^{\mathcal{E}^{(9)}} k^{\mathcal{F}^{(9)}} o^{\tau(y_1,v_1,u_1,h-2)+l}\\ &\qquad = \rho \mathcal{A}_{l1}\{\phi^d \theta^c\} B_{d,A(l,1)}\{\theta^c\} j^{c+x^d A(l,1)} k^{d+L(l,1)} o^l. \end{aligned}$$

*Condition that* $k^{(h)}\gamma = \gamma k$. In (59), replace

$$\mathfrak{H}^{(9)}, \mathfrak{E}^{(9)}, \mathfrak{F}^{(9)} \quad \text{by} \quad \mathfrak{H}^{(10)}, \mathfrak{E}^{(10)}, \mathfrak{F}^{(10)},$$

$$\epsilon_1, y_1, v_1, u_1 \quad \text{by} \quad \epsilon_2, y_2, v_2, u_2,$$

$$\mathcal{A}, A, L \quad \text{by} \quad \mathfrak{D}, R, S.$$

The condition is

$$(60) \qquad \epsilon_2(\chi^{h-1})\,\mathfrak{H}^{(10)} j^{\mathfrak{E}^{(10)}} k^{\mathfrak{F}^{(10)}} o^{\tau(y_2,v_2,u_2,h-2)+l} = \rho \mathfrak{D}_{l1}\{\phi^d \theta^c\} B_{d,R(l,1)}\{\theta^c\} j^{c+x^d R(l,1)} k^{d+S(l,1)} o^l.$$

*Condition that* $o^{(h)}\gamma = \gamma o$. In the left member of (59), replace

$$\mathfrak{H}^{(9)}, \mathfrak{E}^{(9)}, \mathfrak{F}^{(9)} \quad \text{by} \quad \mathfrak{H}^{(11)}, \mathfrak{E}^{(11)}, \mathfrak{F}^{(11)},$$

$$\epsilon_1, y_1, v_1, u_1 \quad \text{by} \quad \epsilon_3, y_3, v_3, u_3.$$

The condition is

$$(61) \qquad \epsilon_3(\chi^{h-1})\,\mathfrak{H}^{(11)} j^{\mathfrak{E}^{(11)}} k^{\mathfrak{F}^{(11)}} o^{\tau(y_3,v_3,u_3,h-2)+l} = \rho(i) j^c k^d o^{l+1}.$$

By [Theorem 1] with $p$ replaced by $h$, we now have the following

THEOREM. *Let an equation,* $f(\xi) = 0$, *be of degree* $pqsh$, *be irreducible in a field* $F$, *and have the roots* (1) *satisfying relations* (2), (3), (4). Consider the algebra over $F$ having the basis $i^m j^n k^t o^l E^r$ $(m \leqq pqsh - 1,$ $n \leqq q - 1,$ $t \leqq p - 1,$ $l \leqq s - 1,$ $r \leqq h - 1)$ with $f(i) = 0$, (5), (6), (7), and

$$E^h = \rho(i) j^c k^d o^l, \qquad \begin{aligned} E^r P(i) &= P(\chi^r) E^r, \\ E^r j^t &= [j^{(r)}]^t E^r, \\ E^r k^t &= [k^{(r)}]^t E^r, \\ E^r o^t &= [o^{(r)}]^t E^r, \end{aligned}$$

where $j^{(r)}$, $k^{(r)}$, $o^{(r)}$ are equal to $\epsilon(\chi^{r-1})\,(J^y K^v O^u)^{(r-2)}$ with subscripts 1, 2, 3, respectively on $\epsilon$, $y$, $v$, $u$, where $J$, $K$, $O$ are as defined in (11). The values of $j^{(r)}$, $k^{(r)}$, $o^{(r)}$ are found by (52), (55), (56) while their $t$-th powers may be found by (26), (29), (31) with altered parameters. This algebra is associative if and only if conditions [6], [14], [15], [18], [45], [49], [54], [61], [69], [72] and the conditions derived from (42), (43), (44), (45), (46), (47), (50), (59), (60), (61) by the application of formulas (39), (41) hold.

We note that these twenty associativity conditions are consistent, since they involve only products of the parameters; thus they are satisfied when all the parameters are equal to unity.

We do not include as conditions the facts that the constants $c_1$, $c_2$, $c_3$,

obtained in each case by the application of formulas (36), (37), (38), are integers, since these are conditions for the existence of the group.[17]

4. *Example. A group of order* 32.

Consider the group of order 32 defined abstractly as follows:[18]

$$\Theta^4 = \Phi^2 = \Psi^2 = X^2 = 1,$$
$$\Theta\Phi = \Phi\Theta, \qquad \Theta X = X\Phi\Theta^3,$$
$$\Theta\Psi = \Psi\Phi\Theta, \qquad \Phi X = X\Phi,$$
$$\Phi\Psi = \Psi\Phi, \qquad \Psi X = X\Psi.$$

This group has an abelian subgroup of order 16, type $(1, 1, 1, 1)$ with generators $\Theta$,[2] $\Phi$, $\Psi$, X.

Let an equation $f(\xi) = 0$ of degree 32 be irreducible in a field $F$ and have the roots

$$\chi^{a_1}\{\psi^{a_2}\phi^{a_3}\theta^{a_4}\} \qquad (a_1 < 2,\ a_2 < 2,\ a_3 < 2,\ a_4 < 4)$$

satisfying the following relations:

$$\theta^4(i) = \phi^2(i) = \psi^2(i) = \chi^2(i) = i,$$
$$\theta\{\phi\} = \phi\{\theta\}, \qquad \theta\{\chi\} = \chi\{\phi\theta^3\},$$
$$\theta\{\psi\} = \psi\{\phi\theta\}, \qquad \phi\{\chi\} = \chi\{\phi\},$$
$$\phi\{\psi\} = \psi\{\phi\}, \qquad \psi\{\chi\} = \chi\{\psi\}.$$

Then the group constants of the general discussion have the values

$$q = 4 \qquad x = 1,$$
$$p = 2 \qquad z_1 = 1,\ w_1 = 1,$$
$$s = 2 \qquad z_2 = 1,\ w_2 = 0,$$
$$h = 2 \qquad u_1 = 0,\ v_1 = 1,\ y_1 = 3,$$
$$e = a = b = c = d = l = 0, \qquad u_2 = 0,\ v_2 = 1,\ y_2 = 0,$$
$$u_3 = 1,\ v_3 = 0,\ y_3 = 0.$$

The constants $c_1$ to $c_{14}$ of Dickson's paper are all zero except

$$c_3 = 2, \qquad c_4 = 1, \qquad c_{11} = 1.$$

The table (32) becomes

17. Dickson, "Construction of Division Algebras," Theorem 6.

18. This group was suggested by Dr. J. K. Senior as one which probably cannot be represented with fewer than four generators.

| Superscript | | $\epsilon_a$ | $y_a$ | $v_a$ | $u_a$ | $\epsilon_b$ | $y_b$ | $v_b$ | $u_b$ | $\mathcal{F}$ | $\mathcal{E}$ | $\mathcal{H}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | | $\epsilon_1$ | 3 | 1 | 0 | $\epsilon_2$ | 0 | 1 | 0 | 2 | 3 | $\epsilon_1\epsilon_2\{\phi\theta^3\}$ |
| (2) | | $\epsilon_2$ | 0 | 1 | 0 | $\epsilon_3$ | 0 | 0 | 1 | 1 | 0 | $\epsilon_2\epsilon_3\{\phi\}$ |
| (3) | | $\epsilon_1$ | 3 | 1 | 0 | $\epsilon_2\epsilon_3(\phi)$ | 0 | 1 | 1 | 2 | 3 | $\epsilon_1\epsilon_2\epsilon_3\{\theta^3\}$ |
| (4) | | $\epsilon_2$ | 0 | 1 | 0 | $\epsilon_3$ | 0 | 0 | 1 | 1 | 0 | $\epsilon_2\epsilon_3\{\phi\}$ |
| (5) | | 1 | 0 | 0 | 0 | $\epsilon_2\epsilon_3(\phi)$ | 0 | 1 | 1 | 1 | 0 | $\epsilon_2\epsilon_3\{\phi\}$ |
| (6) | | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| (7) | 00 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| | 10 | $\epsilon_2$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | $\epsilon_2$ |
| | 01 | 1 | 0 | 0 | 0 | $\epsilon_3$ | 0 | 0 | 1 | 0 | 0 | $\epsilon_3$ |
| (8) | 000 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| | 310 | $\Delta$ | 9 | 3 | 0 | $\epsilon_2$ | 0 | 1 | 0 | 4 | 9 | $\Delta\epsilon_2\{\phi\theta\}$ |
| | 010 | 1 | 0 | 0 | 0 | $\epsilon_2$ | 0 | 1 | 0 | 1 | 0 | $\epsilon_2$ |
| | 001 | 1 | 0 | 0 | 0 | $\epsilon_3$ | 0 | 0 | 1 | 0 | 0 | $\epsilon_3$ |
| (9) | | $\Delta\epsilon_2\{\phi\theta\}$ | 9 | 4 | 0 | $\rho$ | 0 | 0 | 0 | 4 | 9 | $\Delta\epsilon_2\{\phi\theta\}\rho\{\theta\}$ |
| (10) | | $\epsilon_2$ | 0 | 1 | 0 | $\rho$ | 0 | 0 | 0 | 1 | 0 | $\epsilon_2\rho\{\phi\}$ |
| (11) | | $\epsilon_3$ | 0 | 0 | 1 | $\rho$ | 0 | 0 | 0 | 0 | 0 | $\epsilon_3\rho\{\psi\}$ |

where $\Delta \equiv \epsilon_1\epsilon_1\{\phi\theta^3\}[\alpha(i)\alpha(\theta^3)]^2\alpha\{\theta\}\alpha\{\phi\}\alpha\{\theta^2\}\alpha\{\phi\theta^2\}\alpha\{\phi\theta^3\}\epsilon_1\{\theta^2\}$.

After evaluating the exponents, we find the conditions in this paper for the present example. They are

$$\mathcal{H}_{21}(i)j^3k^2o^0 = \alpha(\chi)\mathcal{H}^{(1)}(i)j^3k^2o^0, \tag{42}$$

$$\mathcal{H}_{31}(i)j^3k^4o^1 = \epsilon_1(\chi)\mathcal{H}^{(3)}(i)j^3k^2o^1, \tag{43}$$

$$\mathcal{H}_{32}(i)j^0k^1o^1 = \epsilon_2(\chi)\mathcal{H}^{(5)}(i)j^0k^1o^1, \tag{44}$$

$$\mathcal{G}_1{}^{(4)}(i)j^{12}k^4o^0 = g(\chi), \tag{45}$$

$$\mathcal{G}_2{}^{(2)}(i)j^0k^2o^0 = \beta(\chi)\mathcal{G}_1{}^{(0)}(i)j^0k^0o^0, \tag{46}$$

$$\mathcal{G}_3{}^{(2)}(i)j^0k^0o^2 = \sigma(\chi)\mathcal{H}^{(6)}(i)j^0k^0o^0, \tag{47}$$

$$\rho(i)j^0k^0o^0 = \rho(\chi)\mathcal{H}^{(8)}(i)j^0k^0o^0, \tag{50}$$

$$\epsilon_1(\chi)\mathcal{H}^{(9)}j^0k^4o^0 = \rho\mathcal{A}_{01}B_{0,1}j^1k^0o^0, \tag{59}$$

$$\epsilon_2(\chi)\mathcal{H}^{(10)}j^0k^1o^0 = \rho\mathcal{D}_{01}B_{0,0}j^0k^1o^0, \tag{60}$$

$$\epsilon_3(\chi)\mathcal{H}^{(11)}j^0k^0o^1 = \rho(i)j^0k^0o^1. \tag{61}$$

We may now determine completely the associativity conditions for our algebra.

*Associativity conditions.*

$$[6] \qquad g(i) = g(\theta).$$

$$[14] \qquad \alpha(i)\alpha(\theta)\alpha(\theta^2)\alpha(\theta^3)g(i) = g(\phi).$$

[15] $$\beta(\phi) = \beta(i).$$

[18] $$\alpha(\phi)\alpha(i)\beta(\theta) = \beta(i).$$

[45] $$\delta_1\delta_1(\phi\theta)\delta_1(\theta^2)\delta_1(\phi\theta^3)\alpha(\theta)\alpha(\theta^2)[\alpha(\theta^3)]^2\alpha(\phi\theta^2)\alpha(\phi\theta^3)[\beta(i)]^2 g(i) = g(\psi).$$

[49] $$\alpha(\psi)\delta_1\delta_2(\phi\theta) = \delta_2\delta_1(\phi)\alpha(i).$$

[54] $$\delta_2\delta_2(\phi)\beta(i) = \beta(\psi).$$

[61] $$\sigma(\psi) = \sigma(i).$$

[69] $$\delta_1\delta_1(\psi)\delta_2(\phi\theta)\sigma(\theta)\beta(\theta) = \sigma(i).$$

[72] $$\delta_2\delta_2(\psi)\sigma(\phi) = \sigma(i).$$

(39) $$\epsilon_2\epsilon_1(\phi)\alpha\alpha(\theta)\alpha(\theta^2) = \alpha(\chi)\epsilon_1\epsilon_2(\phi\theta^3).$$

(43) $$\epsilon_3\epsilon_1(\psi)\delta_1\delta_1(\phi\theta)\delta_1(\theta^2)\delta_2(\phi\theta^3)\alpha(\theta)\alpha(\phi\theta^2)\alpha(\theta^2)\beta(\theta^3) = \epsilon_1(\chi)\epsilon_1\epsilon_2\epsilon_3(\theta^3).$$

(45) $$\epsilon_3\epsilon_2(\psi)\delta_2 = \epsilon_2(\chi)\epsilon_2\epsilon_3(\phi).$$

(50) $$\epsilon_1\epsilon_1(\theta^2)\epsilon_1(\phi\theta)\epsilon_1(\phi\theta^3)\alpha\alpha(\phi)\alpha(\phi\theta) \times [\alpha(\theta)]^3[\alpha(\theta^2]^4[\alpha(\theta^3)]^4[\alpha(\phi\theta^3)]^2[\alpha(\phi\theta^2)]^2\beta^2 g^3 = g(\chi).$$

(54) $$\epsilon_2\epsilon_2(\phi)\beta(i) = \beta(\chi).$$

(58) $$\epsilon_3\epsilon_3(\psi)\sigma(i) = \sigma(\chi).$$

(65) $$\rho(i) = \rho(\chi).$$

(76) $$\epsilon_1(\chi)\epsilon_1\epsilon_1(\phi\theta^3)\epsilon_1(\theta^2)\epsilon_2(\phi\theta)\alpha(\theta)\alpha(\phi\theta^2) \times \alpha(\phi\theta^3)\alpha(\phi)\alpha(\theta^2)[\alpha(i)\alpha(\theta^3)\beta(\theta)]^2\rho(\theta)g^2 = \rho(i).$$

(77) $$\epsilon_2\epsilon_2(\chi)\rho(\phi) = \rho(i).$$

(81) $$\epsilon_3\epsilon_3(\chi)\rho(\psi) = \rho(i).$$

If we take $g = \beta = \sigma = \rho = \alpha = \delta_1 = \delta_2 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, the algebra over $F$ determined by our equation, $f(\xi) = 0$, has the basis

$$i^m j^n k^t o^f E^r \qquad (m \leqq 31,\ n \leqq 3,\ t \leqq 1,\ f \leqq 1,\ r \leqq 1),$$

with $f(i) = 0$; and the multiplication table

$$\begin{aligned}
&j^4 = 1, \quad jP(i) = P(\theta)j, \quad j^2P(i) = P(\theta^2)j^2, \quad j^3P(i) = P(\theta^3)j^3,\\
&k^2 = 1, \quad kP(i) = P(\phi)k,\\
&\qquad kj = jk, \quad kj^2 = j^2k, \quad kj^3 = j^3k,\\
&o^2 = 1, \quad oP(i) = P(\psi)o,\\
&\qquad oj = jko, \quad oj^2 = j^2o, \quad oj^3 = j^3ko,\\
&\qquad ok = ko,\\
&E^2 = 1, \quad EP(i) = P(\chi)E,\\
&\qquad Ej = j^3kE, \quad Ej^2 = j^2E, \quad Ej^3 = jkE,\\
&\qquad Ek = kE,\\
&\qquad Eo = oE.
\end{aligned}$$

This algebra is associative.

# LARGE POSITIVE INTEGERS ARE SUMS OF FOUR OR FIVE VALUES OF A QUADRATIC FUNCTION.*

By GORDON PALL.

1. Professor L. E. Dickson † has, in several papers, considered the representation of all positive integers as sums of $s(\geqq 2)$ values of the function

$$mx^2/2 + nx/2 + c, \tag{1}$$

$$(m, n, c \text{ integers};\ m+n \text{ even};\ m > 0).$$

His formulae give actual limits to $p$ in the following theorem of E. Maillet's.‡

*Let $m > 0$, and either: $1°$, $mn$ odd and $(m, n) = 1$; or $2°$, $m$ and $n$ even, $(m+n)/2$ odd, and $(m/2, n/2) = 1$. Then, in case $1°$ every integer $p$ beyond a certain limit* [which he does not give explicitly] *depending only on $m, n$, and $c$ is a sum of four numbers* (1).§ *In case $2°$ the same is true of every sufficiently large odd $p$.*

Since we can replace $p$ by $p - 4c$ we can suppose that $c = 0$. Without further reference in the sequel we shall suppose $m > 0$ and either $1°$ or $2°$. All cases are seen to reduce to these.

Write

$$f(x) = mx^2/2 + nx/2. \tag{2}$$

We discuss for $s = 4$ the solvability of the equation

$$p = f(x_1) + \cdots + f(x_s) \tag{$3_s$}$$

in integers $x_i$; and also in integers $x_i \geqq -k$, $(k \geqq 0)$. If better results exist for $s = 5$ we consider $(3_5)$. The most general theorem obtained is

THEOREM 1. *Except when, in case $2°$, $m/2$ is odd and $n/2$ even,* $(3_4)$

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* National Research Fellow.

† There are five published ones:

I. *American Journal of Mathematics*, Vol. 50 (1928), pp. 1-48.
II. *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 713-720.
III. *Ibid.*, Vol. 34 (1928), pp. 63-72.
IV. *Ibid.*, Vol. 34 (1928), pp. 205-217.
V. *Proceedings of the American Philosophical Society*, Vol. 66 (1927), pp. 281-286.

‡ *Bulletin de la Société Mathématique de France*, Vol. 23 (1895), pp. 40-49.

§ Maillet does not state condition $(12_3)$ below in its true rôle, as leading to integers $x \geqq 0$. He uses "entier" throughout his paper as though he meant "entier $\geqq 0$."

*is solvable in integers* $x_i \geqq 0$ *for every* $p$ *exceeding a certain function of* $m$ *and* $n$. *If* $m/2$ *is odd and* $\geqq 5$, *and* $n/2$ *even, then* $(3_4)$ *is not solvable in integers* $x_i$ *for* 1) *only a finite number of odd* $p > 0$, *and* 2) *an infinite number of even* $p \geqq 0$. *The values* $p$ *in* 2) *are such that* $2mp + n^2$ *forms a finite number, say* $w$, *of progressions*

$$4^{lh}t_v (v = 1, \cdots, w), \quad h = 0, 1, 2, \cdots, \tag{4}$$

*where the* $t_v$ *are even integers and* $l$ *is the least positive integer such that* $2^l \equiv \pm 1 \pmod{m/2}$.

The equation $(3_s)$ is equivalent to

$$8mp + sn^2 = (2mx + n)^2. \tag{$5_s$}$$

Hence our theorems are equivalent to facts about sums of squares of numbers in an arithmetic progression.

For example, for $m/2 = 5$, consider

$$5p + 4j^2 = (5x_1 + j)^2 + \cdots + (5x_4 + j)^2, \tag{$6_j$}$$

where $j = 1$ or 2. We shall see that the only integers $5p + 4 \geqq 0$ for which $(6_1)$ is not solvable are 9, 29, 59, 79, and $4^{2h}t$, where

$$t = 14,\ 24,\ 44,\ 94,\ 184, \quad h \geqq 0. \tag{7}$$

And the only integers $5p + 16 \geqq 0$ for which $(6_2)$ is not solvable in integers $x_i$ are 1, 11, 41, 51, 101, and $4^{2h}t$,

$$t = 6,\ 46,\ 56,\ 176,\ 376, \quad h \geqq 0. \tag{8}$$

Let $j = 1$ or 2. We shall in Section 4 find all integers $q \geqq 0$ such that

$$3q + 4 = (3x_1 + j)^2 + \cdots + (3x_4 + j)^2 \tag{$9_j$}$$

is not solvable in integers $x_i \geqq -k$, $(k \geqq 0)$. It is remarkable that, while there are for any $k \geqq 0$, infinitely many positive $(3q + 4)$'s such that $(9_j)$ is not solvable in integers $x_i \geqq -k$, yet every $3q + 4 \geqq 4$ is a sum of four squares prime to 3.

It is an obvious corollary of Theorem 1 that $(3_5)$ is solvable in integers $x_i$ for every sufficiently large integer $p$. This is seen to be true in integers $x_i \geqq 0$. We tabulate in Sections 3 and 4 a number of the simplest cases $(5_3)$, $(5_4)$, and $(5_5)$.

2. We shall use two extensions of the Cauchy lemma on the solvability in integers $x_i$ of the equations

$$a = x_1^2 + \cdots + x_4^2, \quad b = x_1 + \cdots + x_4. \tag{10}$$

Proofs are given in the writer's paper, "Simultaneous Quadratic and Linear Representation", to appear shortly in the *Quarterly Journal of Mathematics.*

Let $\Lambda$ denote the *form* $4^h(8v+7)$, where $h$ and $v$ are integers, $h \geqq 0$.

LEMMA 1. *Integral solutions* $x_i$ *of* (10) *exist if and only if*

$$a \equiv b \pmod 2, \quad 4a - b^2 \neq \Lambda, \quad 4a \geqq b^2. \tag{11}$$

*None of the* $x_i$ *in* (10) *can be negative if*

$$b \geqq 0, \quad b^2 + 2b + 4 > 3a. \tag{12}$$

*If* $a$, $b$ *are even, solutions in integers* $x_i \geqq 0$ *exist if* $(11_2)$, $(11_3)$, (12), *and merely*

$$3b^2 + 8b + 16 > 8a. \tag{13}$$

LEMMA 2. *If* $s = 5$, 6, *or* 7, *integers* $x_w$ *satisfying*

$$a = x_1^2 + \cdots + x_s^2, \quad b = x_1 + \cdots + x_s \tag{14}$$

*exist if and only if*

$$a \equiv b \pmod 2, \quad sa \geqq b^2. \tag{15}$$

*Such integers* $\geqq 0$ *exist if, in addition to* (15),

$$b \geqq 0, \quad b^2 \geqq 3a - 5.* \tag{16}$$

3. Since $(m, n) = 1$ or 2 every integer $p$ is of the form

$$p = (am + bn)/2, \tag{17}$$

where $a$, $b$ are integers. With Dickson we write $t = (m+n)/2$. Then (17) may be written as

$$p = m(a-b)/2 + tb. \tag{18}$$

LEMMA 3. *Let* $r$ *denote one of the numbers* $0, 1, \cdots, m-1$. *Then*

$$p \equiv rt \pmod m \tag{19}$$

*and* $b \equiv r \pmod m$ *imply that* (17) *defines an integer* $a$ *of the same parity as* $b$. *Conversely, if* $b \equiv r \pmod m$ *and* $a \equiv b \pmod 2$, (17) *implies* (19).

3.1. Taking the simplest case first we find all integers $p$ permitting integer solutions $a$, $b$ of

* If $a$, $b$ are odd, $(16_2)$ may be replaced by $(12_2)$. If $a$, $b$ are even, $(16_2)$ may be replaced by $(11_2)$ and (13).

$$(20) \qquad p = (am + bn)/2, \quad sa \geqq b^2, \quad a \equiv b \pmod 2,$$

where $s$ is given and $\geqq 1$. Conditions $(20_1)$ and $(20_2)$ are given equivalent to $(20_1)$ and

$$(21) \qquad p \geqq g(b)/2, \qquad g(b) = nb + mb^2/s.$$

Let $b_r$ denote the integer $b \equiv r \pmod m$ for which $g(b)$ is least. (By calculus, $2m \mid b_r \mid \leqq m^2 + s \mid n \mid$.) Then the integers required are all $p$ such that

$$(22) \qquad p \geqq g(b_r)/2, \quad p \equiv rt \pmod m, \quad (r = 0, 1, \cdots, m-1).$$

By lemma 2 we have

LEMMA 4. *If $s = 5$, 6, or 7, the values $p$ in (22) form the class of all integers $p$ for which $(3_s)$, or $(5_s)$, is solvable in integers $x_i$.*

We list all cases in which the number of integers $8mp + 5n^2 \geqq 0$ for which $(5_5)$ has no solution in integers $x_i$ is $\leqq 5$, and in which there is not a better result (given later) for $s = 4$.

| | Solvable for every $p \geqq 0$ except the following: |
|---|---|
| $56p + 45 = \sum_5 (14x + 3)^2$, | $p = 1, 3$; |
| $56p + 125 = \sum_5 (14x + 5)^2$, | none ; |
| $5p + 20 = \sum_5 (5x + 2)^2$, | $p = 6, 7, 8, 17$; |
| $16p + 45 = \sum_5 (8x + 3)^2$, | $p = 6$ ; |
| $16p + 5 = \sum_5 (8x + 1)^2$, | $p = 1, 2, 4, 7$; |
| $72p + 125 = \sum_5 (18x + 5)^2$, | $p = 1, 3, 5, 12$; |
| $72p + 245 = \sum_5 (18x + 7)^2$, | $p = 6, 7$. |

3. 2. Second we wish all integers $p$ for which integers $a$, $b$ exist satisfying

$$(23) \qquad (17),\ sa \geqq b^2 \geqq \tau a - \beta b - \gamma, \quad a \equiv b \pmod 2, \quad b \geqq -k,$$

where $\tau$, $\beta$, $\gamma$, $s$, $k$ are given rational numbers, $0 < \tau < s$.

Conditions $(23_1)$ and $(23_2)$ are equivalent to $(23_1)$ and

$$(24) \qquad I_b: \quad g(b)/2 \leqq p \leqq h(b)/2,$$

$$g(b) = nb + mb^2/s, \quad h(b) = nb + m(b^2 + \beta b + \gamma)/\tau.$$

The intervals $I_b$ for $b \geqq -k$ and $\equiv r \pmod m$ give all values $p \equiv rt \pmod m$ for which (23) are solvable.

3. 21. For simplicity we now suppose $n > -m$. Then $g(b)$ and $h(b)$

are monotone increasing as soon as $b \geqq \sigma/2$, $\sigma = \max(s-1,\ \tau-\beta-1)$. For any integer $d \geqq 1$ let $y = y(d)$ be the least integer $\geqq \sigma/2$ such that

$$\boldsymbol{g}(z+d) \leqq h(z) \text{ for every integer } z \geqq y. \tag{25}$$

Then, clearly, every $p$ such that

$$p \geqq \boldsymbol{g}(y+d-1)/2 \tag{26}$$

belongs to at least $d$ consecutive intervals $I_b$.

On completing the square for $z$, (25) becomes

$$[(s-\tau)y - (\tau d - \beta s/2)]^2 \geqq Z, \quad (s-\tau)(y+1/2) \geqq \tau d - \beta s/2, \tag{27}$$

where $Z = s\tau(d^2 - \beta d + \gamma) - s^2(\gamma - \beta^2/4) + s\tau(s-\tau)dn/m$.

3. 22. If $s = 5$, $k = 0$, $\beta = 0$, $\gamma = 5$, $\tau = 3$, then (23) are only sufficient for the solvability of $(5_5)$ in integers $x_i \geqq 0$. It is clear that $(5_5)$ is solvable in integers $x_i \geqq 0$ for every $p$ satisfying (26) with $d = m$. The values $p$ below this limit must be considered separately. This is practicable only for a small $m$; we can make use also of the intervals $I_b$ for $b < y(m) + m - 1$. Also, if $b$ is odd, we can use $(12_2)$ in place of $(16_2)$.

3. 3. Third let $s = 4$ and adjoin the condition $(11_2)$ to (20). If $b_r$ is odd condition $(11_2)$ is superfluous, since, for odd $a$ and $b$, $4a - b^2 \not\equiv \Lambda$.

LEMMA 5. *For a certain $r$ let $b_r$ be even, and let $b'$ denote that one of $b_r \pm m$ for which $\boldsymbol{g}(b')$ is least. If $m$ is odd, or if $m/2$ is even, or if $m/2$ is odd and $p$ such that*

$$mp/2 + n^2/4 \not\equiv 0 \pmod 8, \tag{28}$$

*then $(5_4)$ is solvable in integers $x_i$ for $p \equiv rt \pmod m$ and $\geqq \boldsymbol{g}(b')/2$.*

*Remark.* The only values $p \equiv rt$ and $< \boldsymbol{g}(b')/2$ for which $(5_4)$ can possibly be solvable are in the interval

$$\boldsymbol{g}(b_r)/2 \leqq p < \boldsymbol{g}(b')/2. \tag{29}$$

Since (20) is solvable for these values $p$ only with $b = b_r$ we need merely exclude among these numbers $p$ all for which

$$4(2p - b_r n)/m - (b_r)^2 = \Lambda$$

to obtain all values $p \equiv rt$ and $< \boldsymbol{g}(b')/2$ for which $(5_4)$ is solvable in integers.

If $m$ is odd, $b'$ is odd, and lemma 5 obvious.

If $m/2$ is even, $n/2$ is odd, and one of $a_1 = (2p - nb_r)/m$, $a_2 = (2p - nb')/m$ is $\equiv 2 \pmod 4$. But $a \equiv 2 \pmod 4$, $b$ even, imply $4a - b^2 \neq \Lambda$.

If $m/2$ is odd, $n/2$ is even, and we can choose $k$ uniquely modulo 2 so that

$$m(b_r + km) + 2n = 4Q, \ Q \text{ odd}.$$

Hence, by (17) and (28),

$$m^2(4a - b^2) = 8mp + 4n^2 - (mb + 2n)^2 = 16(8v + 1, 3, \text{or } 5),$$

for one of $b = b_r, b'$. Lemma 5 follows.

In this third case we treat together * the system of equations

$$(30) \qquad mp/2 + n_j^2/4 = \Sigma_i \, (mx_i/2 + n_j/4)^2 \qquad (j = 1, \cdots, l),$$

where $l$ is the least integer $> 0$ such that $2^l \equiv \pm 1 \pmod{m/2}$, and the $n_j/2$ form, apart from sign, a representative set of even residues modulo $m/2$ of $2^h \cdot n/2$ $(h = 0, 1, 2, \cdots)$. For any $h \geqq 0$ and $j$ we can choose a unique $g$ from $(1, \cdots, l)$ so that

$$(31) \qquad 2^h n_g \equiv \pm n_j \pmod m.$$

If, for some $h$ and $j$, $4^{-h}(mp/2 + n_j^2/4)$ is an even integer, every square in the representations of $mp/2 + n_j^2/4$ as a sum of four squares is divisible by $4^h$. Then, by (31),

$$(32) \qquad mp/2 + n_j^2/4 = 4^h(mq/2 + n_g^2/4),$$

where $q$ is an even integer; and for any $x_i$ of $(30_j)$,

$$(33) \qquad mx_i/2 + n_j/4 = 2^h(my_i/2 + n_g/4),$$

where $y_i$ is an integer.

All even values $p$ for which any one of the $l$ equations (30) is solvable in integers $x_i$ can therefore be deduced from a knowledge of all values $p$ satisfying

$$(34) \qquad mp/2 + n_j^2/4 \not\equiv 0 \pmod 8, \ p \geqq -(n_j)^2/2m,$$

for which each equation is solvable.

It is clear, also, that if some equation $(30_g)$ is not solvable in integers $x_i$ for one even value $p = q$ satisfying (34), then each equation $(30_j)$ has

---

* Instead we can consider just the single equation $(5_4)$ for all even values $p$ such that $mp/2 + n^2 \not\equiv 0 \pmod{2.4^l}$.

no solution in integers $x_i$ for infinitely many values $p \geqq 0$, indeed for all $p$ defined by (32) with $h$ as in (31).

We prove for $m/2 \geqq 5$, $m/2$ odd, that $(5_4)$ has no solution in integers $x_i$ for some even $p \geqq -n^2/2m$. We can suppose that $0 < n < m$. Hence the three least squares $(mx_i/2 + n/4)^2$ are the squares of

$$x = n/4,\ y = m/2 - n/4,\ z = m/2 + n/4;\ x^2 < y^2 < z^2.$$

Their even sums by four possibly $\leqq z^2 + 3x^2$ are

$$4x^2,\ 2y^2 + 2x^2,\ 4y^2,$$

three in number. But at least four even values $p$ satisfy

$$0 \leqq pm/2 + n^2/4 \leqq z^2 + 3x^2.$$

The results for (6) may be verified in about ten minutes by means of (22) with $m = 10$, $n = 4$ or 8, whence $t = 7$ or 9; and $b_r = 1,\ 3,\ -5,\ -3,\ -1$, $(r = 1,\ 3,\ 5,\ 7,\ 9)$; and by means of lemma 5 and succeeding remark with $(b_r, b') = (0, -10)$, $(2, -8)$, $(4, -6)$ or $(-6, 4)$, $(-4, 6)$, $(-2, 8)$ $(r = 0, 2, 4, 6, 8)$.

The only integers $p \geqq 0$ for which equation is not solvable in integers $x$ are

$$\begin{aligned}
40p + 4 &= \Sigma_4(10x+1)^2, && p = 1 \quad ;\\
40p + 36 &= \Sigma_4(10x+3)^2, && \text{none} \quad ;\\
56p + 4 &= \Sigma_4(14x+1)^2, && p = 1, 2, 5 \quad ;\\
56p + 36 &= \Sigma_4(14x+3)^2, && p = 1, 3, 25 \quad ;\\
56p + 100 &= \Sigma_4(14x+5)^2, && p = 5, 23 \quad ;\\
16p + 4 &= \Sigma_4(8x+1)^2, && p = 1, 2, 4, 7, 30;\\
16p + 36 &= \Sigma_4(8x+3)^2, && p = 5, 6, 26.
\end{aligned}$$

3.4. Fourth we adjoin $(11_2)$ to (23) for $s = 4$. From the intervals $I_b$ must be excluded all values $p$ such that

$$4(2p - bn)/m - b^2 = \Lambda.$$

As in 3.3, we see that, if $m$ is odd, or $m/2$ even, or if $m/2$ is odd and $p$ satisfies (28), then $(5_4)$ is solvable in integers $x_i \geqq 0$ if (26) holds with $d = 2m$, $\beta = 2$, $\gamma = \tau = 3$.

4. For $s = 3$ we note that each of the equations

$$8p + 3 = \Sigma_3(4x+1)^2,\quad 24p + 3 = \Sigma_3(6x+1)^2,$$

is solvable in three integers $x$ for every $p \geqq 0$. Not as trivial is the fact that

*every positive multiple* $3N$ *of* 3 *is a sum of three squares prime to* 3, *if* $3N \neq \Delta$. This is evident unless $3N$ is also a multiple of 9. Then it follows from the identity

$$9(x^2+y^2+z^2) = (2x+2y-z)^2 + (2z+2x-y)^2 + (2y+2z-x)^2.$$

For any $p \geqq 0$ we can write $3p+4 = 4^w(3q+4)$, where $q \not\equiv 0 \pmod 4$ and $q \geqq 0$, or $q = 0$. Write $3N = 3q$ unless $3N = \Delta$, then $3N = 3q+3$. Add 4 or 1 respectively. Hence: *every* $3p+4$ *is a sum of four squares prime to* 3, one of the squares being 1 or 4. It is of interest to recall Pepin's formula for the number of representations in $x^2+9y^2+9z^2+9t^2$. A corollary is that if $p$ is even, the number of representations of $3p+4$ as a sum of four squares prime to 3 is equal to $16\zeta'_1(3p+4)$, sixteen times the sum of the odd divisors of $3p+4$.

To discuss $(9_j)$ for integers $x_i \geqq 0$ we use the formulae of Section 3 with $m=6$, $n=4j$ $(j=1$ or $2)$, $s=4$, $q=p+4(j-1)$. If $d=6$, $(27_1)$ becomes

$$(35) \qquad (y-14)^2 \geqq 292+48j; \text{ or } (y-8)^2 \geqq 121+32j;$$

according as

$$(36) \qquad \tau=\gamma=3,\ \beta=2, \text{ or } \tau=\beta=2^2_3,\ \gamma=5.$$

Hence $y = 32+j$, $20+j$ respectively, and, by (26), every $p \geqq g(y+5)/2$ belongs to at least six intervals $I_b$. If $p$ is odd or double of an odd this means the existence of a suitable $a$ which is odd or double of an odd. Hence $(9_j)$ is solvable in integers $x_i \geqq 0$ for every $p$ such that

$$(37) \qquad p \geqq 1021+138j,\ p \text{ odd}; \quad p \geqq 466+96j,\ p \equiv 2 \pmod 4.$$

Setting $m=6$, $t=3+2j$ in Dickson's table I * we get a list $T_j$ of all sums $\leqq 1335+28j$ of four values of $f(x)$ for integers $x \geqq 0$. This was checked independently to $489+44j$ by setting $m=6$, $t=1$ in Dickson's table III.† From $T_1$ and $T_2$ we get

LEMMA 6. *The following are all odd* $q>0$ *such that*

$$(38) \qquad 3q+4 \neq \Sigma_4(1^2, 4^2, 7^2, 10^2, \cdots):$$

$$(39) \quad q = 1, 3, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 35, 39, 41, 45, 47, 51, 55, 57, 63, 67, 69, 73, 75, 79, 83, 91, 97, 103, 107, 109, 113, 119,$$

* *L. c.* II, pp. 718, 719.
† *L. c.* IV, pp. 207, 208.

129, 131, 137, 143, 147, 149, 159, 165, 183, 189, 195, 211, 235, 239, 247, 263, 275, 305, 321, 339, 345, 403, 509, 585, 643.

*The following are all odd $q > 0$ such that*

(40) $$3q + 4 \neq \sum_4(2^2, 5^2, 8^2, 11^2, \cdots):$$

(41) $q = 1, 3, 5, 7, 9, 13, 15, 17, 19, 21, 23, 27, 29, 33, 35, 37, 39, 41, 47, 49, 53, 55, 59, 61, 65, 67, 69, 73, 79, 81, 85, 87, 91, 93, 97, 101, 105, 111, 117, 123, 125, 129, 131, 135, 137, 149, 151, 155, 161, 167, 169, 173, 181, 187, 205, 211, 217, 223, 229, 237, 261, 269, 273, 279, 285, 293, 305, 335, 341, 355, 373, 379, 409, 441, 551, 631, 693, 1169.$

LEMMA 7. *All positive integers $q \equiv 2 \pmod 4$ satisfying* (38) *are*

(42) $q = 2, 6, 14, 18, 22, 30, 34, 46, 50, 58, 62, 74, 78, 86, 102, 114, 126, 142, 154, 206, 258, 270, 334, 398.$

*All such satisfying* (40) *are*

(43) $q = 2, 6, 10, 14, 22, 26, 30, 34, 42, 46, 54, 62, 66, 74, 78, 86, 94, 98, 110, 118, 130, 142, 154, 162, 174, 198, 230, 254, 286, 298, 366, 434.$

For $q \equiv 0 \pmod 8$ we use $d = 12$ which is effective, since an integer $a \equiv 4 \pmod 8$ will appear from $p = 3a + 2jb$ for one of $b \equiv 2, 6, 10 \pmod{12}$. Under $(36_2)$, $(27_1)$ yields

$$(y_{12} - 20)^2 \geqq 673 + 64j, \quad y_{12} = 47 + j.$$

Hence $(9_j)$ is solvable in integers $x_i \geqq 0$ for all $p \equiv 4 - 4j \pmod 8$ and $\geqq 2524 + 208j$. Now $I_{58}$, $I_{46}$, $I_{34}$ are respectively

$2523 \leqq p - 116j \leqq 3964$, $1587 \leqq p - 92j \leqq 2524$, $867 \leqq p - 68j \leqq 1408$;
$I_{54}$, $I_{42}$ are $2187 \leqq p - 108j \leqq 3448$, $1323 \leqq p - 84j \leqq 2116$;
$I_{50}$, $I_{38}$ are $1875 \leqq p - 100j \leqq 2968$, $1083 \leqq p - 76j \leqq 1744$.

Above the limit $1335 + 28j$ of $T_j$ and outside the intervals $I_b$ just given we find that there remain just the following values

$p \equiv b(3 + 2j) \pmod 6$ and $\equiv 4(1 - j) \pmod 8$:

| | |
|---|---|
| $p = 1428 + 68j + 24v$ | $(v = 0, 1, \cdots, 6 + j);$ |
| $2532 + 92j + 24v$ | $(v = 0, j - 1);$ |
| $2124 + 84j + 24v$ | $(v = 0, 1, \cdots, 2 + j);$ |
| $1332 + 36j + 24v$ | $(v = 0, \cdots, 2j - 1);$ |
| $1764 + 76j + 24v$ | $(v = 0, 1, \cdots, 4 + j).$ |

All these numbers were located in the extension of $T_j$ obtained by setting $m = 6$, $t = 3 + 2j$ in Dickson's table on pp. 13, 14 of I, except $p = 2624$ if $j = 1$, and $p = 2716$ and 2740 if $j = 2$. But, $2624 = f(29) + f(3) + 2f(1)$, $2716 = f(27) + f(11) + 2f(1)$, $2740 = f(27) + 2f(7) + f(5)$.

Hence table $T_j$ gives

LEMMA 8. *All integers* $q \geqq 0$, $\equiv 0 \pmod 8$, *and satisfying* (40), *are*

(44) 8, 24, 40, 80, 96, 160, 200, 216, 424, 520, 1224.

*All integers* $q \geqq 0$, $\equiv 0 \pmod 8$, *and satisfying* (40), *are*

(45) 0, 8, 16, 40, 48, 56, 72, 80, 104, 112, 144, 168, 192, 224, 256, 280, 416, 440.

From lemma 6 we can verify

LEMMA 9. *All odd* $q > 0$ *such that* $(9_1)$ *is not solvable in* $x_1 \geqq -1$ *are*

(46) 9, 13, 25, 29, 41, 45, 47, 69, 75, 79, 97, 109, 149, 165, 189, 235, 305, 509.

*For each of these* $(9_1)$ *is solvable in integers* $x_i \geqq -2$. *All odd* $q > 0$ *such that* $(9_2)$ *is not solvable in integers* $x_i \geqq -1$ *are*

(47) 5, 7, 13, 15, 19, 27, 35, 39, 47, 53, 55, 79, 85, 91, 93, 111, 123, 167, 187, 211, 223, 261, 279, 285, 335, 551; 33, 59, 129.

*For the numbers preceding the semicolon* $(9_2)$ *is solvable in integers* $x_i \geqq -2$. *For* $q = 33, 59, 129$ *it is first solvable in integers* $x_i \geqq -3$.

Consider the class of the sixty-nine numbers $t = 3q + 4$, where $q$ runs over the even integers $q$ of (42), (43), (44), and (45). This is the set of all even numbers $\equiv 1 \pmod 3$ and $\not\equiv 0 \pmod 8$, such that the list of all representations

$$t = 3q + 4 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad \text{all } y_i \text{ prime to } 3, \quad y_1 \geqq y_2 \geqq y_3 \geqq y_4 > 0, \tag{48}$$

does not contain two representations $(y_i)$ and $(z_i)$ such that each $y_i \equiv 1$ and each $z_i \equiv 2 \pmod 3$.

If $k \geqq 0$ and $q$ even, the equation

$$4^k(3q + 4) = (3x_1 + j)^2 + \cdots + (3x_4 + j)^2 \tag{49}$$

will not be solvable in integers $x_i \geqq -k$ if and only if in every representation (48),

(50) some $y_i$ satisfies $2^k y_i > 3k - j$ and $2^k y_i \not\equiv j \pmod 3$.

For else we can choose $x_i$ from $3x_i + j = \pm 2^h y_i$.

An application of (50) to the list of all representations (48) for each $t$ gives all values $h$ for which (49) is not solvable in integers $x_i \geqq -k$. For example, for

(51) $t = 4, 34, 52, 130, 148, 172, 202, 286, 298, 316, 340, 358, 394, 436, 490, 526, 580, 598, 676, 694, 766, 772, 844, 862, 898, 1102, 1252, 1306,$

the values $q = (t-4)/3$ appear in (43) or (45) and not in (42) or (44). Hence each $t$ possesses a representation (48) with each $y_i \equiv 1$ and none with each $y_i \equiv 2 \pmod 3$. To exclude this representation, (50) requires $2^h \not\equiv j \pmod 3$, that is, $h \equiv j \pmod 2$. Also, (51) possess respectively the partitions $(y_1, y_2, y_3, y_4)$ which follow:

$(1,1,1,1)$, $(5,2,2,1)$, $(5,5,1,1)$, $(11,2,2,1)$, $(11,5,1,1)$, $(11,5,5,1)$, $(14,2,1,1)$, $(14,8,5,1)$, $(17,2,2,1)$, $(17,5,1,1)$, $(17,5,5,1)$, $(17,8,2,1)$, $(14,14,1,1)$, $(17,11,5,1)$, $(20,8,5,1)$, $(20,11,2,1)$, $(23,5,5,1)$, $(23,8,2,1)$, $(23,11,5,1)$, $(20,17,2,1)$, $(26,8,5,1)$, $(23,11,11,1)$, $(29,1,1,1)$, $(26,11,8,1)$, $(26,14,5,1)$, $(29,14,8,1)$, $(35,5,1,1)$, $(29,20,8,1)$,

in which every $y_i > 1$ is $\equiv 2 \pmod 3$. Since we already have $2 \cdot 2^h \equiv j \pmod 3$, (50) requires finally that

(52) $$2^h > 3k - j, \quad h \equiv j \pmod 2;$$

and it is evident that (52) serves to assure (50) for every further representation (48) of any $t$ in (51).

In some cases it is necessary to consider more than one representation (48) of $t$, and sometimes all representations. If

(53) $t = 58, 154, 178, 292, 310, 346, 382, 604, 622, 778, 814, 1006, 1198, 1276, 3676,$

then $q$ belongs to (42) or (44), and not to (43) or (45). Also they have the respective partitions

$(7,2,2,1)$, $(10,7,2,1)$, $(13,2,2,1)$, $(16,4,4,2)$, $(16,7,2,1)$, $(13,13,2,2)$, $(19,4,2,1)$, $(22,10,4,2)$, $(19,16,2,1)$, $(25,10,7,2)$, $(25,13,4,2)$, $(28,13,7,2)$, $(28,19,7,2)$, $(34,10,4,2)$, $(52,22,22,2)$,

in which each $y_i \not\equiv 2$ is $\equiv 1 \pmod 3$. The condition on $h$ is seen to be

(54) $$2 \cdot 2^h > 3k - j, \quad h \not\equiv j \pmod 2.$$

If $t$ possesses a representation (48) with one or more $y_i = 1$ and the remaining $y_i \equiv 2$ and also a representation with one or more $y_i = 2$ and the remaining $y_i \equiv 1 \pmod 3$, then (50) demands

(55) $\quad 2^h > 3k - j$, or $2 \cdot 2^h > 3k - j \geqq 2^h$ and $h \not\equiv j \pmod 2$.

If also $q$ belongs to both (42) and (43), or (44) and (45), then (55) assures (50) for all remaining representations. The values $t$ for which these properties hold are

(56) $\quad t = 10, 28, 70, 124, 190, 226, 262, 430, 466,$

the respective partitions being

$(2, 2, 1, 1)$; $(5, 1, 1, 1) = (4, 2, 2, 2)$; $(8, 2, 1, 1) = (7, 4, 2, 1)$;
$(11, 1, 1, 1) = (10, 4, 2, 2)$; $(13, 4, 2, 1) = (11, 8, 2, 1)$; $(14, 5, 2, 1)$
$= (13, 7, 2, 2)$; $(16, 2, 1, 1) = (14, 8, 1, 1)$; $(20, 5, 2, 1) = (19, 7, 4, 2)$;
$(20, 8, 1, 1) = (19, 10, 2, 1)$.

All representations (48) of 17 values $t$:

$94 = (8, 5, 2, 1) = (7, 5, 4, 2)$;
$244 = (14, 4, 4, 4) = (13, 7, 5, 1) = (13, 5, 5, 5) = (11, 11, 1, 1)$
$= (11, 7, 7, 5) = (10, 8, 8, 4)$;
$22 = (4, 2, 1, 1)$;
$106 = (10, 2, 1, 1) = (8, 5, 4, 1) = (7, 7, 2, 2) = (7, 5, 4, 4)$;
$238 = (14, 5, 4, 1) = (13, 8, 2, 1) = (13, 7, 4, 2) = (11, 10, 4, 1)$
$= (11, 8, 7, 2) = (10, 8, 7, 5)$;
$46 = (5, 4, 2, 1)$;
$142 = (11, 4, 2, 1) = (10, 5, 4, 1) = (8, 7, 5, 2)$;
$82 = (8, 4, 1, 1) = (7, 5, 2, 2) = (7, 4, 4, 1) = (5, 5, 4, 4)$;
$166 = (11, 5, 4, 2) = (10, 8, 1, 1) = (10, 7, 4, 1) = (10, 5, 5, 4)$
$= (8, 7, 7, 2)$;
$220 = (14, 4, 2, 2) = (13, 7, 1, 1) = (13, 5, 5, 1) = (11, 7, 7, 1)$
$= (11, 7, 5, 5) = (10, 10, 4, 2)$;
$334 = (17, 5, 4, 2) = (16, 7, 5, 2) = (14, 11, 4, 1) = (14, 8, 7, 5)$
$= (13, 10, 8, 1) = (13, 10, 7, 4) = (11, 10, 8, 7)$;
$76 = (8, 2, 2, 2) = (7, 5, 1, 1) = (5, 5, 5, 1)$;
$484 = (20, 8, 4, 2) = (19, 11, 1, 1) = (19, 7, 7, 5) = (17, 13, 5, 1)$
$= (17, 11, 7, 5) = (16, 14, 4, 4) = (16, 10, 8, 8) = (13, 13, 11, 5)$
$= (11, 11, 11, 11)$;
$652 = (25, 5, 1, 1) = (23, 11, 1, 1) = (23, 7, 7, 5) = (22, 10, 8, 2)$
$= (19, 17, 1, 1) = (19, 13, 11, 1) = (19, 11, 11, 7) = (17, 17, 7, 5)$

$= (17, 13, 13, 5) = (17, 11, 11, 11) = (16, 14, 14, 2) = (16, 14, 10, 10)$
$= (14, 14, 14, 8)$;

1564 $= (38, 10, 4, 2) = (37, 13, 5, 1) = (37, 11, 7, 5) = (35, 17, 7, 1)$
$= (35, 17, 5, 5) = (35, 13, 13, 1) = (35, 13, 11, 7) = (34, 20, 2, 2)$
$= (34, 14, 14, 4) = (31, 23, 7, 5) = (13, 19, 11, 11) = (31, 17, 17, 5)$
$= (29, 25, 7, 7) = (29, 23, 13, 5) = (29, 19, 19, 1) = (28, 26, 10, 2)$
$= (28, 22, 14, 10) = (26, 26, 14, 4) = (26, 22, 20, 2) = (25, 25, 17, 5)$
$= (25, 23, 19, 7) = (25, 23, 17, 11) = (25, 19, 17, 17)$
$= (22, 22, 20, 14)$;

508 $= (22, 4, 2, 2) = (20, 10, 2, 2) = (19, 11, 5, 1) = (19, 7, 7, 7)$
$= (17, 13, 7, 1) = (17, 13, 5, 5) = (17, 11, 7, 7) = (14, 14, 10, 4)$
$= (13, 13, 13, 1) = (13, 13, 11, 7)$;

1324 $= (35, 7, 7, 1) = (35, 7, 5, 5) = (34, 10, 8, 2) = (32, 14, 10, 2)$
$= (32, 10, 10, 10) = (31, 19, 1, 1) = (31, 17, 7, 5) = (31, 13, 13, 5)$
$= (31, 11, 11, 11) = (29, 19, 11, 1) = (29, 17, 13, 5) = (26, 22, 10, 8)$
$= (26, 16, 14, 14) = (25, 25, 7, 5) = (25, 23, 13, 1) = (25, 23, 11, 7)$
$= (25, 19, 17, 7) = (25, 19, 13, 13) = (25, 17, 17, 11)$
$= (22, 22, 16, 10)$.

Applying (50) and collecting the preceding results we have

THEOREM 2. *Let $j = 1$ or 2, $k \geqq 0$. The odd values $q$ for which $(9_j)$ has no solution in integers $x_i \geqq -k$ are seen in lemmas 6 and 9. The even values $q$ are $(4^h t - 4)/3$ where:*

1) *$t$ is given by (51); $h$ by (52);*
2) *$t$ by (53); $h$ by (54);*
3) *$t$ by (56); $h$ by (55);*
4) *$t = 94, 244$; $2^h > 3k - j$, or $5 \cdot 2^h > 3k - j \geqq 2^h$ and $h \not\equiv j \pmod 2$;*
5) *$t = 22, 106, 238$; $2 \cdot 2^h > 3k - j$, or $4 \cdot 2^h > 3k - j \geqq 2 \cdot 2^h$ and $h \equiv j$;*
6) *$t = 46, 142$; $4 \cdot 2^h > 3k - j$, or $5 \cdot 2^h > 3k - j \geqq 4 \cdot 2^h$ and $h \not\equiv j$;*
7) *$t = 82, 166, 220, 334$; $4 \cdot 2^h > 3k - j$, $h \equiv j \pmod 2$;*
8) *$t = 76, 484, 652, 1564$; $5 \cdot 2^h > 3k - j$, $h \not\equiv j$;*
9) *$t = 508, 1324$; $7 \cdot 2^h > 3k - j$, $h \equiv j$.*

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# ON STIELTJES CONTINUED FRACTIONS.

By J. SHOHAT (JACQUES CHOKHATE).

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The object of this paper is to investigate the relation of the denominators of the *odd* convergents of the continued fraction

$$\int_a^b \frac{d\psi(y)}{x-y} = \frac{1}{/l_1 x} + \frac{1}{/l_2} + \frac{1}{/l_3 x} + \frac{1}{/l_4} + \cdots$$

to the corresponding system of orthogonal and normal Tchebycheff polynomials, and to show their application to the theory of mechanical quadratures.

1. Let $\psi(x)$ be a bounded non-decreasing function, defined on the—finite or infinite—interval $(a, b)$, such that

$$\textit{all moments } \alpha_i = \int_a^b x^i d\psi(x) \quad (i = 0, 1, \cdots) \textit{ exist, with } \alpha_0 > 0.$$

We have then a corresponding system of orthogonal and normal Tchebycheff polynomials

$$(1) \qquad \begin{gathered} \phi_n(x) \equiv \phi_n(x; a, b; d\psi) \equiv \phi_n(x; d\psi) = a_n x^n + \cdots \quad (n = 0, 1, \cdots) \\ a_n \equiv a_n(d\psi) > 0, \end{gathered}$$

uniquely determined by means of the relations

$$(2) \qquad \int_a^b \phi_m(x)\phi_n(x)\,d\psi(x) = 0 \ (m \neq n),\ 1 \ (m = n) \quad (m, n = 0, 1, \cdots).$$

We shall use also the orthogonal polynomials

$$(3) \qquad \begin{gathered} \Phi_n(x) \equiv \Phi_n(x; a, b; d\psi) \equiv \Phi_n(x; d\psi) \equiv \phi_n(x)/a_n = x^n + \cdots \\ (n = 0, 1, \cdots), \end{gathered}$$

which, by virtue of (1), are uniquely determined by means of

$$(4) \qquad \int_a^b \phi_n(x)\,G_{n-1}(x)\,d\psi(x) = 0,$$

where $G_s(x) = \sum_{i=0}^{s} g_i x^i$ stands in general for an arbitrary polynomial of degree $\leq s$ (subject, in some cases, to certain explicitly stated conditions).

The polynomials (3), are, as is known, the denominators of the successive convergents of the "associated" continued fraction *

$$(5)\qquad \int_a^b \frac{d\psi(y)}{x-y} = \frac{\lambda_1/}{/x-c_1} - \frac{\lambda_2/}{/x-c_2} - \cdots \qquad (\lambda_i, c_i = \text{const.}),$$

which always exists, $\psi(x)$ being of the above type. (5), in turn, is intimately connected with the "corresponding" continued fraction *

$$(6)\quad \int_a^b \frac{d\psi(y)}{x-y} = \frac{1/}{/l_1 x} + \frac{1/}{/l_2} + \frac{1/}{/l_3 x} + \frac{1/}{/l_4} + \cdots \equiv \frac{b_1/}{/x} - \frac{b_2/}{/1} - \frac{b_3/}{/x} - \frac{b_4/}{/1} - \cdots \dagger$$

$$\left((l_i,\ b_i = \text{const.},\ b_i = -\frac{1}{l_{i-1}l_i}\right),$$

which exists, if $a \geqq 0$. In fact, denoting by $A_n(x)/B_n(x)$ $(n = 0, 1, \cdots)$ the successive convergents to (6), we have (1):

$$(7)\qquad B_{2n+\epsilon}(x) = l_1 l_2 \cdots l_{2n+\epsilon} x^{n+\epsilon} + \cdots \quad (B_{2n+1}(0) = 0) \qquad (\epsilon = 0, 1)$$

$$(8)\qquad \int_a^b B_{2n+\epsilon}(x) x^k d\psi(x) = 0 \qquad (k = 0, 1, \cdots, n-1)$$

$$(9)\qquad \int_a^b B_{2n}^2(x)\, d\psi(x) = \frac{1}{l_{2n+1}}\,; \quad \int_a^b x\left(\frac{B_{2n+1}(x)}{x}\right)^2 d\psi(x) = -\frac{1}{l_{2n+2}}\,.$$

(8), with $\epsilon = 0$, shows, by virtue of (3, 4, 9), that

$$(10)\qquad B_{2n}(x) \equiv \Phi_n(x;\ d\psi)/\Phi_n(0;\ d\psi) \equiv \phi_n(x;\ d\psi)/\phi_n(0;\ d\psi),$$

for

$$(11)\qquad l_{2n+1} = \Phi_n^2(0;\ d\psi)\dagger \qquad (n = 0, 1, \cdots).$$

*Hence, the denominators of the even convergents of* (6) *coincide with the denominators of the convergents of* (5), which, therefore, can be obtained from the former by "contraction."

*We wish now to find a simple expression for* $B_{2n+1}(x)$ *in terms of the orthogonal and normal polynomials* (1). (8), with $\epsilon = 1$, is equivalent to

$$(12)\qquad \int_a^b B_{2n+1}(x) G_{n-1}(x)\, d\psi(x) = 0,$$

* O. Perron, ***Die Lehre von den Kettenbrüchen*** (1913), Ch. IX. This author considers $\int_a^b \frac{d\psi(y)}{x+y}$. It follows that the even $l-s$ in (6) differs in sign from the corresponding quantities used by Perron.

† Jacques Chokhate, "Sur le développement de l'intégrale $\int_a^b \frac{p(y)\,dy}{x-y}$ en fraction continue et sur les polynomes de Tchebycheff," ***Rendiconti di Palermo***, Vol. 47 (1923), pp. 25-46.

which, combined with (7, 3, 4), yields:

$$(13)\quad B_{2n+1}(x) = l_1 l_2 \cdots l_{2n+1}\left[\Phi_{n+1}(x) - \Phi_n(x)\frac{\Phi_{n+1}(0)}{\Phi_n(0)}\right] \quad (n = 0, 1, \cdots),$$

i. e. $B_{2n+1}(x)$ is a linear combination of $\phi_n(x)$ and $\phi_{n+1}(x)$. This result is due to A. Markoff.*

It is our aim to derive from it a more interesting and important expression of $B_{2n+1}(x)$. For this we shall make use of Darboux's formulae, also of the polynomials $\phi_n(x; a, b; x d\psi)$.

2. Consider the relation †

$$(14)\quad \begin{aligned} G_n(x) &= \int_a^b K_n(x, t)\, G_n(t)\, d\psi(t) \\ K_n(x, t) &\equiv \sum_{i=0}^{n} \phi_i(x)\phi_i(t). \end{aligned}$$

Replacing here $G_n(t)$ by $(t - x)G_{n-1}(t)$, we get:

$$(15)\quad 0 = \int_a^b (t - x) K_n(x, t)\, G_{n-1}(t)\, d\psi(t),$$

which, combined with (2), gives:

$$(16)\quad (t - x)K_n(x, t) = A_{n+1}(x)\phi_{n+1}(t) + A_n(x)\phi_n(t).$$

Compare in (16) the coefficients of $t^{n+1}$ and then interchange $x$ and $t$:

$$A_{n+1}(x) = \frac{a_n}{a_{n+1}}\,\phi_n(x), \qquad A_n(x) = -\frac{a_n}{a_{n+1}}\,\phi_{n+1}(x).$$

Thus we get Darboux's formulae: ‡

$$(17)\quad K_n(x, t) = \frac{a_n}{a_{n+1}}\left[\frac{\phi_n(x)\phi_{n+1}(t) - \phi_n(t)\phi_{n+1}(x)}{t - x}\right].$$

$$(18)\quad \begin{aligned} K_n(x) \equiv K_n(x;\, d\psi) \equiv K_n(x, x) &\equiv \sum_{i=0}^{n} \phi_i{}^2(x) \\ &= \frac{a_n}{a_{n+1}}\left[\phi'_{n+1}(x)\phi_n(x) - \phi'_n(x)\phi_{n+1}(x)\right]. \end{aligned}$$

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* A. Markoff, "Note sur les fractions continues," *Bulletin de l'Académie Impériale des Sciences de Russie*, (5), (1895), Vol. 2, pp. 9-16.

† J. Shohat, "On a General Formula in the Theory of Tchebycheff Polynomials and Its Applications," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 569-583.

‡ Darboux, "Mémoire sur l'approximation des fonctions de très grands nombres," *Journal des Mathématiques* (3), Vol. 4 (1878), pp. 5-56, 377-416. This process of deriving Darboux's formulae is due, in part, to M. J. Geronimus.

3. We assume now that

$$x \leq a, \quad \text{or} \quad \geq b. \tag{19}$$

With such $x$ we get readily, comparing (15) and (2):

$$K_n(x,t) = C_n(x)\phi_n(t; \mid t-x \mid d\psi),$$

and comparing the coefficients of $t^n$:

$$K_n(x,t) = [a_n(d\psi)/a_n(\mid t-x \mid d\psi)]\,\phi_n(x; d\psi)\cdot\phi_n(t; \mid t-x \mid d\psi) \tag{20}$$
$$(x \geq a, \leq b)$$

$$K_n(x) = [a_n(d\psi)/a_n(\mid t-x \mid d\psi)]\cdot\phi_n(x; d\psi)\cdot\phi_n(x; \mid t-x \mid d\psi) \tag{20 bis}$$

$$K_n(0,t) = [a_n(d\psi)/a_n(td\psi)]\,\phi_n(0; d\psi)\phi_n(t; td\psi) \qquad (a \geq 0) \tag{21}$$

$$K_n(0) = [a_n(d\psi)/a_n(td\psi)]\phi_n(0; d\psi)\phi_n(0; td\psi) \tag{21 bis}$$
$$= (b_{2n+2})^{1/2}\phi_n(0; d\psi)\phi_n(0; td\psi).$$

*Formulae* (20, 21) *are of great importance in the theory of orthogonal Tchebycheff Polynomials.*

4. *Hereafter* $a \geqq 0$. Making use of *

$$x\phi_n(x; xd\psi) = (b_{2n+3})^{1/2}\phi_{n+1}(x; d\psi) + (b_{2n+2})^{1/2}\phi_n(x; d\psi) \tag{22}$$

$$= -\phi_{n+1}(0; d\psi)/\phi_n(0; d\psi) = (b_{2n+2}/b_{2n+3})^{1/2} \tag{23}$$

$$l_{2n} = -1/b_{2n+1}l_{2n+1}; \quad \lambda_n = b_{2n-2}b_{2n-1} = a^2{}_{n-2}/a^2{}_{n-1}; \tag{24}$$
$$b_{2n+2} = a_n{}^2(d\psi)/a_n{}^2(xd\psi),$$

we get:

$$l_{2n} = \frac{1}{a^2{}_{n-1}\Phi_{n-1}(0)\Phi_n(0)} = \frac{1}{(\lambda_{n+1})^{1/2}\phi_{n-1}(0)\phi_n(0)}. \tag{25}$$

We put now $x = 0$ in (15) and compare with (12). Since $B_{2n+1}(0) = 0$, we have necessarily

$$\int_a^b [B_{2n+1}(t) - CtK_n(0,t)]\,G_{n-1}(t)\,d\psi(t) = 0, \tag{26}$$

where the constant $C$ can be so chosen that $B_{2n+1}(t)/t - CK_n(0,t)$ becomes a polynomial *of degree* $n-1$. Identifying it with $G_{n-1}(t)$ in (26), we get, making use of (21, 24), *the fundamental result:*

$$B_{2n+1}(t)/t = CK_n(0,t) = D(b_{2n+2})^{1/2}\phi_n(0, d\psi)\phi_n(t; td\psi),$$

which gives, by virtue of (9, 23, 25), $D = 1$, and *we thus get the desired expression of* $B_{2n+1}(x)$ *in an unexpectedly simple form:*

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* *Loc. cit.* (†).

$$(27) \qquad B_{2n+1}(x) \equiv (b_{2n+2})^{\frac{1}{2}} x\phi_n(0;\, d\psi)\phi_n(x;\, xd\psi) \equiv xK_n(0, x).$$

Hence, *the study of the denominators* $B_{2n+1}(x)$ *is reduced to that of* $\phi_n(x;\, xd\psi)$, which, for example, in the case of polynomials of Jacobi or Laguerre are orthogonal polynomials of the same class respectively.

5. We proceed to derive an application to mechanical quadratures. It is known * that if in the formula of mechanical quadratures

$$(28) \qquad \int_0^b f(x)\,d\psi(x) = \sum_{i=0}^{n} H_i f(x_i) + R_n(f)$$

$$0 = x_0 < x_1 < \cdots < x_n < b; \qquad \phi(x) = \prod_{i=0}^{n} (x - x_i),$$

$$H_i = \int_0^b \frac{\phi(x)}{(x - x_i)\phi'(x_i)}\, d\psi(x)$$

we wish to have $R_n(f) \equiv 0$ for $f(x) \equiv G_{2n}(x)$, we must choose $x_1, x_2, \cdots, x_n$ as zeros of $\phi_n(x;\, xd\psi)$. Formula (27) leads to the following

THEOREM. *The formula of mechanical quadratures* (28), *with* $n+1$ *ordinates, one of which is fixed at* $x_0 = 0$, *holds true* (i. e. $R_n \equiv 0$) *for an arbitrary polynomial of degree* $\leq 2n$, *if and only if the* $n+1$ *ordinates correspond to the zeros of the polynomial* $B_{2n+1}(x)$. Moreover, $H_0 = 1/K_n(0;\, d\psi)$.

The first part of our statement is proved by (27). In order to establish the second part, we write [see (21)]:

$$H_0 = \int_0^b \frac{x\phi_n(x;\, xd\psi)}{x(x\phi_n(x;\, xd\psi))'_{x=0}}\, d\psi(x) = \int_0^b \frac{\phi_n(x;\, xd\psi)}{\phi_n(0;\, xd\psi)}\, d\psi(x) =$$

$$= \int_0^b \frac{K_n(0;\, x)\,d\psi(x)}{K_n(0;\, d\psi)} = \frac{1}{K_n(0;\, d\psi)}.$$

This result is of importance in connection with the *practical application* of the formula (28).

We thus obtained in a general form a result previously established,† in an entirely different way, for a very special class of Jacobi polynomials.

As to the value of $K_n(0;\, d\psi)$, it can be found by means of (21 bis), or even still simpler in some special cases. As an illustration Laguerre polynomials may serve. Here

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* J. Shohat, "On a Certain Formula of Mechanical Quadratures with Non-Equidistant Ordinates," *Transactions of the American Mathematical Society*, Vol. 31 (1930), pp. 448-463.

† *Loc. cit.* (*).

$$(a, b)=(0, \infty), \qquad d\psi(x)= e^{-x}dx,$$

$$K_n(0)=-e^{-x}K_n(x)\Big|_0^{\infty}=-\int_0^{\infty}[e^{-x}K_n(x)]'dx$$

$$=\int_0^{\infty}e^{-x}K_n(x)\,dx-\int_0^{\infty}e^{-x}K'_n(x)\,dx=n+1.$$

[making use of (2)].

6. For the sake of simplicity we take $b=1$. For $n$ very large we can derive, in a very simple manner, an upper limit for $H_0$, for the general class of Tchebycheff polynomials corresponding to

$$d\psi(x)=p(x)\,dx, \quad p(x)=x^{\sigma}q(x) \qquad (\sigma>-1,\ q(0)\neq 0), \tag{29}$$

$q(x)$ continuous at $x=0$. We make use of the fact that *

$$1/K_n(0)=\text{min.}\int_0^1 G_n^2(x)\,d\psi(x), \quad \text{with } G_n(0)=1. \tag{30}$$

Taking here $G_n(x)\equiv(1-x)^n$, we get:

$$H_0=1/K_n(0)<\int_0^1(1-x)^{2n}p(x)\,dx=\int_0^1 x^{2n}p_1(x)\,dx \tag{31}$$

$$(p_1(x)\equiv p(1-x)).$$

On the other hand, (29) leads to †

$$\int_0^1 x^{2n}p_1(x)\,dx=\frac{\Gamma(\sigma+1)\,q(0)+o(1)}{(2n)^{\sigma+1}} \qquad (n\to\infty). \tag{32}$$

Hence,

$$H_0=1/K_n(0)=O(1/n^{\sigma+1}) \qquad \text{(under condition (29))}. \tag{33}$$

*Remark.* The above results hold, mutatis mutandis, for the upper endpoint $(x=b)$ of the interval $(0, b)$ in question.

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* *Loc. cit.* (†).

† J. Shohat, "On the Asymptotic Expressions of Certain Definite Integrals," *Annals of Mathematics*, Vol. 27 (1925), pp. 3-11.

# ON A PROBLEM OF M. J. SHOHAT.

By J. GERONIMUS.

In his paper "On a General Formula in the Theory of Tchebycheff Polynomials and Its Applications," * M. J. Shohat considers the following problem: *Given an interval* $(a, b)$ *finite or infinite, a function* $p(x)$ *non-negative on this interval, for which all integrals*

$$\int_a^b p(x)x^\nu dx, \qquad (\nu = 0, 1, 2, \cdots)$$

*exist, find the minimum* $L$ *of the integral* $\int_a^b p(x)y^2(x)dx$, *where* $y(x)$ *is a polynomial of degree* $\leqq n$, $y(x) = \sum_{i=0}^{n} \sigma_i x^i$, *subject to the condition* $\omega(y) \equiv \sum_{i=0}^{n} \alpha_i \sigma_i = a$ $(a \neq 0)$ $\alpha_i$ *being given real numbers.*

In solving this problem M. Shohat considers certain polynomials which may be regarded as a generalization of orthogonal polynomials. It is the object of the present paper to generalize the problem of M. Shohat, assuming that the coefficients of our polynomial $y(x)$ are connected by $m+1$ linear relations. Some properties of these polynomials will be considered.

1. *Find the minimum value* $L$ *of the integral*

$$\int_a^b p(x)y^2(x)dx \tag{1}$$

*under the following conditions:*

$$\omega_i(y) \equiv \sum_{s=0}^{n} \alpha_{is}[y^{(s)}(\eta)/s!] = d_i \qquad (i = 0, 1, 2, \cdots, m), \tag{2}$$

$\eta$, $d_i$ *and* $\alpha_{is}$ $(i = 0, 1, 2, \cdots, m;\ s = 0, 1, 2, \cdots, n)$ *being given real numbers.*

Introduce $m+1$ polynomials $u_i(x)$ of the $n$-th degree possessing the property

$$\int_a^b p(x)u_i(x)G_n(x)dx = \omega_i(G_n) \qquad (i = 0, 1, 2, \cdots, m), \tag{3}$$

* *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 569-583.

where $G_n(x)$ is an *arbitrary* polynomial of degree $\leqq n$. For $\eta = 0$ we obtain the polynomials introduced by M. A. Angelesco and M. J. Shohat, which are a generalization of orthogonal polynomials.* Put

$$y(x) = \sum_{i=0}^{n} \gamma_i u_i(x) + z(x),$$

where $\gamma_i$ are to be found from the conditions $\omega_k(z) = 0$ $(k = 0, 1, 2, \cdots, m)$. It is always possible to determine the $\gamma_i$ because the determinant of the above system of equations

$$a = \| a_{is} \| \qquad (i, s = 0, 1, 2, \cdots, m),$$

where

$$a_{is} = \int_a^b p(x) u_i(x) u_s(x)\, dx, \tag{4}$$

is positive, being the discriminant of the positive definite quadratic form

$$\sum_{i,s=0}^{m} a_{is}\gamma_i\gamma_s = \int_a^b p(x)\{\sum_{s=0}^{m} \gamma_s u_s(x)\}^2 dx.$$

*We shall show that the condition of minimum of* (1) *is:* $z(x) \equiv 0$. In fact, by virtue of the relations $\omega_i(z) = 0$ $(i = 0, 1, \cdots, n)$,

$$\int_a^b p(x)y^2(x)\,dx = \int_a^b p(x)y_1^2(x)\,dx + \int_a^b p(x)z^2(x)\,dx \geqq \int_a^b p(x)y_1^2(x)\,dx$$

$$\left(y_1(x) = \sum_{s=0}^{m} \gamma_s u_s(x)\right),$$

which proves our statement.

Thus we see that the polynomial $y(x)$, for which the minimum of (1) is attained, may be found from the equation

$$\begin{vmatrix} y(x) & u_0(x) & u_1(x) & \cdots & u_m(x) \\ d_0 & a_{00} & a_{01} & \cdots & a_{0m} \\ d_1 & a_{10} & a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ d_m & a_{m0} & a_{m1} & \cdots & a_{mm} \end{vmatrix} = 0. \tag{5}$$

We find in a similar way that the minimal value $L$ may be obtained from the equation:

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* A. Angelesco, "Sur les polynomes orthogonaux en rapport avec d'autres polynomes," *Bulletin de la Société des Sciences de Cluj*, t. I (1921), pp. 44-59.

$$(6)\qquad \begin{vmatrix} L & d_0 & d_1 & \cdot & \cdot & \cdot & d_m \\ d_0 & a_{00} & a_{01} & \cdot & \cdot & \cdot & a_{0m} \\ d_1 & a_{10} & a_{11} & \cdot & \cdot & \cdot & a_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_m & a_{m0} & a_{m1} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix} = 0.$$

2. We can readily express $u_i(x)$ and $a_{is}$ $(i, s = 0, 1, \cdots, m)$ in terms of orthogonal normal Tchebycheff polynomials $\phi_k(x)$ $(k = 0, 1, 2, \cdots)$, corresponding to the given characteristic function $p(x)$ and the interval $(a, b)$. Using their fundamental property

$$(7)\qquad \int_a^b p(x)\phi_i(x)\phi_s(x)\,dx = 0 \ (i \neq s),\ 1 \ (i = s),$$

we find easily from (3) and (4) that

$$(8)\qquad \begin{aligned} u_i(x) &= \sum_{t=0}^{n} \omega_i(\phi_r)\phi_r(x), \\ a_{is} &= \sum_{t=0}^{n} \omega_i(\phi_r)\omega_s(\phi_r). \end{aligned} \qquad (i, s = 0, 1, 2, \cdots, m).$$

We can also give the solution of our problem in terms of the polynomials of Appell corresponding to the given characteristic function $p(x)$ and the interval $(a, b)$.* Put

$$(9)\qquad c_k \equiv c_k(\eta) = \int_a^b p(x)\,(x - \eta)^k dx \qquad (k = 0, 1, 2, \cdots).$$

Then our minimal problem is reduced to minimizing the quadratic form

$$\sum_{t,s=0}^{n} a_i a_s c_{i+s}$$

under the conditions

$$\omega_k(y) = \sum_{s=0}^{n} \alpha_{ks} a_s = d_k, \qquad (k = 0, 1, 2, \cdots, m).$$

Using the classical method we find readily:

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* P. Appell, "Sur une classe de polynomes," *Annales de l'École Normale Supérieure*, t. 9 (1880), pp. 119-144.

$$(10)\qquad \begin{vmatrix} y(x) & 1 & x-\eta & \cdots & (x-\eta)^n & 0 & 0 & \cdots & 0 \\ 0 & c_0 & c_1 & \cdots & c_n & \alpha_{00} & \alpha_{10} & \cdots & \alpha_{m0} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & c_n & c_{n+1} & \cdots & c_{2n} & \alpha_{0n} & \alpha_{1n} & \cdots & \alpha_{mn} \\ d_0 & \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ d_m & \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{mn} & 0 & 0 & \cdots & 0 \end{vmatrix} = 0$$

$$(11)\qquad \begin{vmatrix} L & 0 & 0 & \cdots & 0 & d_0 & d_1 & \cdots & d_m \\ 0 & c_0 & c_n & \cdots & c_n & \alpha_{00} & \alpha_{10} & \cdots & \alpha_{m0} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & c_n & c_{n+1} & \cdots & c_{2n} & \alpha_{0n} & \alpha_{1n} & \cdots & \alpha_{mn} \\ d_0 & \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ d_m & \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{mn} & 0 & 0 & \cdots & 0 \end{vmatrix} = 0.$$

To find an expression for the $u_i(x)$ and $a_{is}$, we put in (3)

$$G_n(x) = (x-\eta)^k \qquad (k = 0, 1, 2, \cdots, n),$$

which yields:

$$(12)\qquad \begin{vmatrix} u_i(x) & 1 & x-\eta & \cdots & (x-\eta)^n \\ \alpha_{i0} & c_0 & c_1 & \cdots & c_n \\ \alpha_{i1} & c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_{in} & c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} = 0$$

$$(i = 0, 1, 2, \cdots, m),$$

$$(13)\qquad \begin{vmatrix} a_{ik} & \alpha_{k0} & \alpha_{k1} & \cdots & \alpha_{kn} \\ \alpha_{i0} & c_0 & c_1 & \cdots & c_n \\ \alpha_{i1} & c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_{in} & c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} = 0$$

$$(i, k = 0, 1, \cdots, m).$$

3. Hereafter,

$$(14)\qquad \omega_i(G_n) \equiv G_n^{(i)}(\eta) \qquad (i = 0, 1, 2, \cdots, m).$$

We see from (4) that

$$(15)\qquad a_{ik} = \sum_{s=0}^{n} \phi_r^{(i)}(\eta)\,\phi_r^{(k)}(\eta) = (-1)^{i+k} i!\, k!\, C_n^{ik}/C_n$$

$$(i, k = 0, 1, \cdots, m),$$

where $C_n = \| c_{ik} \|$ $(i, k = 0, 1, \cdots, n)$, $c_{ik} = c_{i+k}$ and $C_n^{ik}$ is the minor of this determinant corresponding to the element $c_{ik}$. This formula leads to some interesting results involving persymmetric determinants formed by the polynomials of Appell.

$$(16)\qquad \begin{vmatrix} H_1(z) & H_2(z) & \cdots & H_n(z) \\ H_2(z) & H_3(z) & \cdots & H_{n+1}(z) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ H_n(z) & H_{n+1}(z) & \cdots & H_{2n-1}(z) \end{vmatrix} = \frac{2^{n(n-1)/2} 1!\,2! \cdots (n-1)!}{i^{n^2}} \cdot H_n(iz),$$

where

$$(17)\qquad H_k(x) = (-1)^k e^{x^2} (d^k/dx^k) e^{-x^2} \qquad (k = 0, 1, 2, \cdots)$$

is the Hermite polynomial of degree $k$. Similarly,

$$(18)\qquad \begin{vmatrix} P_1(z) & P_2(z) & \cdots & P_n(z) \\ P_2(z) & P_3(z) & \cdots & P_{n+1}(z) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ P_n(z) & P_{n+1}(z) & \cdots & P_{2n-1}(z) \end{vmatrix} = \left(\frac{(z^2-1)^{n^2/2}}{2^{n^2-n}}\right) \cdot T_n\left(\frac{z}{(z^2-1)^{1/2}}\right),$$

where $P_k(z)$ is the Legendre polynomial and

$$(19)\qquad T_k(y) = \frac{[y + (y^2-1)^{1/2}]^k + [y - (y^2-1)^{1/2}]^k}{2}$$

is the trigonometric polynomial.*

*Suppose further that* $m = 0$ *and*

$$(20)\qquad \omega_0(G_n) \equiv \omega(G_n) \equiv G_n(c).$$

Then, putting

$$\phi_k(x) = \sum_{s=0}^{k} d_s^{(k)} x^{k-s} \qquad (k = 0, 1, 2, \cdots),$$

we see that the polynomial

$$(21)\qquad u(x) \equiv \sum_{k=0}^{n} \phi_k(x)\phi_k(c) = \frac{d_0^{(n)}}{d_0^{(n+1)}} \cdot \frac{\phi_{n+1}(c)\phi_n(x) - \phi_n(c)\phi_{n+1}(x)}{c - x}$$

possesses the property

$$(22)\qquad \int_a^b p(x)u(x)G_n(x)\,dx = G_n(c).\dagger$$

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* J. Geronimus, *On Some Persymmetric Determinants Formed by Polynomials of M. Appell.* (In press.)

† *Loc. cit.* (*).

Taking, for instance, $c=1$, $-a=b=1$, $G_n(x)=(x-z)^n$, we see that

$$(23) \qquad (1-z)^n=\int_{-1}^{1} p(x)\,(x-z)^n u(x)\,dx.^*$$

Returning to an arbitrary real value of $c$ not belonging to the interval $a<x<b$, we see from (21) that $u(x)$ is identical with the orthogonal polynomial corresponding to the interval $(a, b)$ and the characteristic function $p(x)\,(c-x)$. Hence, the following identity:

$$(24) \qquad \begin{vmatrix} c_0 & c_1 & \cdots & c_{n+1} \\ c_1 & c_2 & \cdots & c_{n+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n} \\ 1 & c & \cdots & c^{n+1} \\ 1 & x & \cdots & x^{n+1} \end{vmatrix} = \begin{vmatrix} c_0 & c_1 & \cdots & c_n & 1 \\ c_1 & c_2 & \cdots & c_{n+1} & c \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} & c^n \\ 1 & x & \cdots & x^n & 0 \end{vmatrix} \cdot (c-x),$$

*where the c's are arbitrary numbers.*

4. Suppose that

$$a_{ik}\equiv\sum_{s=0}^{n}\phi_s^{(i)}(\eta)\,\phi_s^{(k)}(\eta), \qquad (i, k=0, 1, \cdots, m)$$

is expressible as

$$(25) \qquad a_{ik}=\alpha_i^{(n)}\alpha_k^{(n)}h_{ik},$$

where $\alpha_i^{(n)}$ depends on $n$ and $i$ and $h_{ik}$ depends on $i$ and $k$ only. Such is the case, for instance, for Legendre polynomials,† where

$$(26) \quad a_{ik} = \frac{(n+i+1)(n+k+1)P_n^{(i)}(\eta)P_n^{(k)}(\eta)+(1-\eta^2)P_n^{(i+1)}(\eta)P_n^{(k+1)}(\eta)}{2(i+k+1)},$$

if $\eta=+1, -1, 0$. Under the condition (25), we have

$$(27) \qquad \frac{A_m^{ik}}{A_m}=\frac{1}{\alpha_i^{(n)}\alpha_k^{(n)}}\cdot\frac{H_m^{ik}}{H_m},$$

where

$$A_m=\|\,a_{ik}\,\|, \qquad H_m=\|\,h_{ik}\,\| \qquad (i, k=0, 1, 2, \cdots, m),$$

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* *Loc. cit.* (†).

† J. Geronimus, "Sur l'écart moyen quadratique minimal de zéro d'un polynome dans l'intervalle donné" (in Russian), *Transactions of the Kharkow Mathematical Society*, Série 4, t. 2 (1928), p. 17.

and $A_m^{ik}$, $H_m^{ik}$ resp. are the corresponding minors of these determinants. In a similar manner we may write

$$\frac{\bar{A}_m^{ik}}{\bar{A}_m} = \frac{1}{\alpha_i^{(m)}\alpha_k^{(m)}} \cdot \frac{H_m^{ik}}{H_m}, \tag{28}$$

where $\bar{A}_m = \| \bar{a}_{ik} \|$ $(i, k = 0, 1, \cdots, m)$ and

$$\bar{a}_{ik} = \sum_{r=0}^{m} \phi_r^{(i)}(\eta)\phi_r^{(k)}(\eta). \tag{29}$$

Thus we obtain:

$$\frac{A_m^{ik}}{A_m} = \frac{\alpha_i^{(m)}\alpha_k^{(m)}}{\alpha_i^{(n)}\alpha_k^{(n)}} \cdot \frac{\bar{A}_m^{ik}}{\bar{A}_m}.$$

Using (15) and the properties of minors of reciprocal determinants,* we obtain the following formula:

$$\frac{\bar{A}_m^{ik}}{\bar{A}_m} = (-1)^{i+k} \cdot \frac{c_{ik}}{i!\,k!},$$

and finally,

$$\frac{A_m^{ik}}{A_m} = \frac{\alpha_i^{(m)}\alpha_k^{(m)}}{\alpha_i^{(n)}\alpha_k^{(n)}} \cdot \frac{(-1)^{i+k}}{i!\,k!}\, c_{ik} \qquad (i, k = 0, 1, \cdots, m). \tag{30}$$

Returning to our minimal problem and using (6), we obtain the following result.

*The minimal value $L$ of the integral $\int_a^b p(x)y^2(x)dx$, being given $y^{(s)}(\eta) = d_s$ $(s = 0, 1, 2, \cdots, m)$, may be written thus [when (25) holds true]:* $L = \int_a^b p(x)y_1^2(x)dx$, $y_1(x) = \sum_{i=0}^{m} \frac{\alpha_i^{(m)}}{\alpha_i^{(n)}} \cdot \frac{(x-\eta)^i}{i!}\, d_i$.

*Example.* $p(x) = 1$, $-a = b = 1$, $\eta = \pm 1$. Here

$$y_1(x) = \sum_{i=0}^{m} \frac{(m+i+1)P_m^{(i)}(\eta)}{(n+i+1)P_n^{(i)}(\eta)} \cdot \frac{(x-\eta)^i}{i!}\, d_i.$$

The same formula is valid for $\eta = 0$, but in this case we must take $m-1$ (resp. $n-1$) instead of $m$ (resp. $n$), when $m-i$ (resp. $n-i$) is odd.

* E. Pascal, *I determinanti* (1923), p. 52.

# ON THE CRITICAL POINTS OF NON-DEGENERATE NEWTONIAN POTENTIALS.

By TSAI-HAN KIANG.

## I. INTRODUCTION.

1. *Definitions.** In this paper an application will be made of a theorem of Morse on the relations between the critical points of a real function of $n$ independent variables (Theorem 5, Morse I, pp. 387-388) to the proof of the existence of critical points, or equilibrium points as better known in physics, of a class of Newtonian potentials.†

We shall use the terminology of analysis situs as defined by Alexander,‡ substituting the terms *cycle* and *Betti number* for *closed chain* and *connectivity number* respectively. Terms in analysis situs not found in Alexander's paper will be as defined in Lefschetz's *Colloquium Lectures on Topology.* Throughout the present paper the terms *Betti number* and *manifold* will be used as abbreviations of *Betti number, mod* 2 and *orientable manifold* respectively.

Let $f(x)$ be a real single-valued function of three variables $(x) = (x_1,$

* The following works will be cited repeatedly in this paper. Hereafter each of these works will be referred to by the name of its author followed by a Roman numeral when necessary.

Kellogg: O. D. Kellogg, *Foundations of Potential Theory,* Berlin (1929); Morse I: M. Morse, "Relations Between the Critical Points of a Real Function of $n$ Independent Variables," *Transactions of American Mathematical Society,* Vol. 30 (1925), pp. 345-396; Morse II: M. Morse, "The Analysis and Analysis Situs of Regular $n$-Spreads in $(n + r)$-space," *Proceedings of the National Academy of Sciences,* Vol. 12 (1927), pp. 813-817.

† The method of the present paper has been extended by the author in another paper, "On the Existence of Critical Points of Green's Functions for Three-Dimensional Regions," which will be published shortly elsewhere. In the latter the regions considered are of very general types from the point of view of analysis situs. Both of these papers are parts of the author's thesis at Harvard University, accepted in June, 1930. Since they were practically completed, there has appeared a paper by J. J. Gergen, "Mapping of a General Type of Three-Dimensional Region on a Sphere," *American Journal of Mathematics,* Vol. 52 (1930), pp. 197-224. In his paper Gergen showed that any Green's function for a three-dimensional region bounded by a torus has a critical point.

‡ J. W. Alexander, "Combinatorial Analysis Situs," *Transactions of the American Mathematical Society,* Vol. 28 (1926), pp. 301-329.

$x_2, x_3$). A point will be called a *critical point* of the function $f$ if it is a zero of all the three partial derivatives of the first order of $f$. The value the function $f$ assumes at a critical point will be called a *critical value* of $f$. A critical point will be called *degenerate* or *non-degenerate* according as it is or is not a zero of the hessian of the function. A function without degenerate critical points will be called a *non-degenerate* function.

Suppose $f$ is of class $C^2$ * in a neighborhood of a non-degenerate critical point $(x^0)$. For simplicity, we suppose $(x^0) = (0)$. Let $f^0_{ij}$ be the partial derivative of $f$ of second order with respect to $x_i$ and $x_j$ evaluated at $(0)$. By a real non-singular linear transformation the non-singular quadratic form $f^0_{ij}x_ix_j$ † can be reduced to one of the forms:

$$\pm x_1^2 \pm x_2^2 \pm x_3^2.$$

The number of negative signs is called the *type number* of the critical point $(0)$ of $f$ by Morse in Morse I. There are four types of critical points.

2. *Morse's theorem.* Suppose a real, single-valued, non-degenerate function $f(x)$ is defined in a closed bounded region $R$ of the real 3-space of three real variables $(x) = (x_1, x_2, x_3)$, and the function $f(x)$ is of class $C^2$ in the interior of $R$. We shall assume that $f$ in $R$ fulfills the following boundary conditions.

(1) The partial derivatives of $f$ of the first and second orders take on continuous boundary values on the boundary $B$ of $R$.

(2) The boundary $B$ of $R$ consists of two sets, $B'$ and $B''$, of closed regular surfaces of class $C^2$.‡ The value of $f$ on all the surfaces of $B'$ is equal to a constant $c'$ and the value of $f$ on all the surfaces of $B''$ is equal to a constant $c''$, where $c'$ is greater but $c''$ is less than the value of $f$ at any interior point of $R$.

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* A function is said to be of class $C^n$, if it is continuous together with all its partial derivatives of orders $1, 2, \cdots, n$. The hypothesis of $C^3$ of $f$ in a neighborhood of the critical point in Morse I has been shown since by Morse to be unnecessary.

† We adopt the convention in tensor analysis under which a repetition of a subscript in a product means that the product is to be summed for admissible values of the subscript.

‡ A regular surface is a closed, bounded, and connected set of points the coördinates of points neighboring any given point of which can be represented by three functions of class $C^1$ of two parameters, in such a way that the jacobians of the functions with respect to the two parameters do not vanish simultaneously. If, moreover, the functions are of class $C^n$, $n > 1$, the surface is said to be of class $C^n$.

(3) At any point of $B$ the value of the outer normal derivative $f_n$ of $f$ with respect to $R$ * does not vanish.

Under all the above conditions the function $f$ will be said to be *an admissible function in R.*

The region $R$ can be shown to be a 3-complex.† From (2) and (3), at any point of $B'$ the value of the outer normal derivative $f_n$ of $f$ is positive and at any point of $B''$ the value of $f_n$ is negative. A surface of $B'$ or of $B''$, as characterized by this statement, will be called *a surface of positive or negative type with respect to f and R.*

From a discussion of the trajectories orthogonal to the manifolds $f =$ constant one can show that the region $R'$ of points in $R$ satisfying

$$f \leqq c'' + e^2,$$

$e$ being a sufficiently small positive number, has the same Betti numbers as the manifold $f = c''$. See Morse I, pp. 355-360. From Theorem 5, Morse I, pp. 387-388, as applied to the present regions $R'$ and $R$, we have the following theorem.

THEOREM A. *Let $f(x)$ be an admissible function in a region $R$. Let $M_k$ $(k = 0, 1, 2, 3)$ be the number of critical points of the kth type in the region. Let $R_k$ and $R_k'$ be the kth Betti numbers of the complexes $R$ and $B''$ respectively. Then there exist integers $M_k^+$ and $M_k^-$ of which $M_3^+$ and $M_0^-$ are zero and the rest are positive or zero such that*

$$M_k = M_k^+ + M_k^-, \quad (k = 0, 1, 2, 3), \tag{2.1}$$

*and such that*

$$R_k - R_k' = M_k^+ - M_{k+1}^-, \quad (k = 0, 1, 2). \tag{2.2}$$

3. *New boundary conditions.* In our investigation we do not meet with exactly the boundary conditions of § 2, but the following ones.

(1a) The boundary $B$ of $R$ consists of two sets, $B_1$ and $B_2$, of closed regular surfaces. Each surface of the set $B_1$ is of class $C^2$. On such a surface the value of $f$ is equal to a constant and the partial derivatives of $f$ of the first and second orders take on continuous boundary values.

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* By the outer normal derivative $f_n$ at a point of $B$ with respect to $R$ we mean the unilateral directional derivative of $f$ at that point along the normal to $B$ in the sense that leads from interior points of $R$ to the boundary.

† S. S. Cairns, "The Cellular Structure and Approximation of Regular Spreads," *Proceedings of the National Acaedmy of Sciences*, Vol. 16 (1930), pp. 488-491.

(2a) Each surface of the set $B_2$ is of class $C^3$. On such a surface the partial derivatives of $f$ of the first order take on continuous boundary values.

(3a) The boundary surfaces may also be groupped into two sets, $B'$ and $B''$, such that every surface of $B'$ is of positive type with respect to $f$ and $R$ and every surface of $B''$ is of negative type (§ 2).

Under this set of new boundary conditions instead of that of § 2, the function $f$ will be called *an admissible function in $R$ under the new boundary conditions.*

These new boundary conditions also imply that the region $R$ is a 3-complex and that $f$ has no critical point on $B$.

On comparing this set of new boundary conditions with that set of § 2, one finds that an admissible function in $R$ under the new boundary conditions with $B_2$ as an empty set of points differs from an admissible function in $R$ of § 2 in that the former does not necessarily take on the absolute maximum on and only on all the boundary surfaces of positive type and the absolute minimum on and only on all the boundary surfaces of negative type. However we shall prove the following theorem.

THEOREM B. *Theorem A holds for an admissible function $f$ in $R$ under the new boundary conditions.*

The proof of this theorem consists in the reduction of the new boundary conditions to those of § 2. The reduction will be carried out in the following two steps in accordance with the remark on the difference of the two sets of boundary conditions made above.

*First step.* It can be shown that, by a redefinition of $f$ in $R$ in a neighborhood of each surface of $B_2$ (cf. Morse I, pp. 394-396), we shall obtain a new function $\bar{f}$ in a new region $\bar{R}$ which fulfills the new boundary conditions with $\bar{B}_2$ * as an empty set of points and which has the following properties: (a) The surfaces $B_1$ are still among the boundary surfaces of $\bar{R}$, and $\bar{R}$ is a closed subregion of $R$, homeomorphic to $R$. (b) The two sets, $\bar{B}'$ and $\bar{B}''$, of boundary surfaces of positive and negative types with respect to $\bar{f}$ and $\bar{R}$ are respectively homeomorphic to the two sets, $B'$ and $B''$, of boundary surfaces of positive and negative types with respect to $f$ and $R$. (c) The critical points of $\bar{f}$ are identical in position and type with those of $f$.

*Second step.* Now, suppose a function $f$ in $R$ fulfills the new boundary

* The set $\bar{B}_2$ for $\bar{f}$ and $\bar{R}$ is taken to have the same meaning as the set $B_2$ for $f$ and $R$. This convention will be followed in this section.

conditions with $B_2$ as an empty set of points. We shall show that by a redefinition of $f$ in $R$ in a neighborhood of $B$ in a similar but simpler way, we can obtain an admissible function $\bar{f}$ in the same region $R$ in the sense of § 2, with the following properties: $(a_1)$ A boundary surface of $B$ of positive or negative type with respect to $f$ and $R$ is of the same type with respect to $\bar{f}$ and $R$. $(b_1)$ The critical points of $\bar{f}$ are identical in position and type with those of $f$.

Let us fix our attention on one surface $S$ of $B'$, on which $f = d$, say. Since no critical point of $f$ is on $S$, through each point of $S$ there is one and only one orthogonal trajectory. We can represent each orthogonal trajectory in such a way that the parameter $t$ at any point equals the value of $f$ at that point (Morse I, p. 358). The parameter $t$ for each trajectory is equal to $d$ on $S$. It is possible to choose a constant $t_1 < d$ so that the points on all the trajectories for which $t_1 \leqq t \leqq d$ constitute a neighborhood $N$ of $S$, in which there are no critical points of $f$ and hence through any point of $N$ there is one and only one such trajectory. Let $t_2$ be another constant between $t_1$ and $d$. For those points in $N$ for which $t_1 \leqq t \leqq t_2$, set

$$\bar{f}(x) \equiv f(x),$$

and for those points in $N$ for which $t_2 \leqq t \leqq d$, set

$$\bar{f}(x) \equiv f(x) + K(f(x) - t_2)^4,$$

where $K$ is a sufficiently large positive constant. Then we find at points in $N$ for which $t_2 \leqq t \leqq d$,

$$\bar{f}_i \equiv f_i(1 + 4K(f - t_2)^3),$$

$$\bar{f}_{ij} \equiv f_{ij}(1 + 4K(f - t_2)^3) + f_i f_j \cdot 12K(f - t_2)^2.$$

Hence, in the interior of $R$ the function $f$ is still of class $C^2$, and on $S$ its partial derivatives of the first and second orders take on continuous boundary values. From the relations between $\bar{f}_i$ and $f_i$ and the fact that $4K(f - t_2)^3 \geqq 0$ at points in $N$ for which $t_2 \leqq t \leqq d$, $f$ has no critical point in $N$. On $S$,

$$\bar{f} = d + K(d - t_2)^4,$$

an arbitrarily large constant. Then $S$ is a surface of positive type with respect to $\bar{f}$ and $R$.

Similar argument can be applied to any boundary surface $T$ of the set $B''$. In this case we shall obtain a new function with the same properties

except that this new function equals an arbitrarily small constant on $T$ and that $T$ is a surface of negative type with respect to this new function and $R$.

If we redefine $f$ in neighborhoods of all the boundary surfaces simultaneously, and choose a suitable constant $K$ for each surface in the redefinition, the new function obtained is admissible in $R$ in the sense of § 2, has the property $(a_1)$, and has its critical points identical in position with those of $f$. Since the type of a critical point is defined with reference to a small neighborhood of the critical point, and since the new function and the original function $f$ are identical in a small neighborhood of each critical point, the new function has the property $(b_1)$ also.

This completes the reduction of the new boundary conditions to those of § 2. Hence our theorem is proved.

## II. The Newtonian Potential Due to a Non-Degenerate admissible Volume, Simple, or Double Distribution.

4. *The potential.* Suppose in the 3-space there are $s$ distinct closed regular surfaces $E_k (k = 1, 2, \cdots, s)$ of class $C^3$, no two of which enclose a common point. Let $D$ designate the closed infinite region bounded by these surfaces and let $E$ designate the sum of the surfaces $E_k$.

We shall consider a point at infinity added to the 3-space, so that the extended space becomes homeomorphic to the 3-sphere.

Suppose there is a volume distribution of matter of piecewise continuous density in the closed finite regions bounded by $E$, or a simple distribution of matter of density satisfying uniformly a Hölder's condition (Kellogg, p. 152) over $E_k$, or a double distribution of moment of class $C^2$ over $E_k$. In the case of double distribution, it should be understood that the potential and its partial derivatives of the first order at any point $P$ of $E$ are defined by their limiting values at $P$ as $P$ is approached from the interior points of $D$. The potential due to the volume, simple, or double distribution is therefore analytic in the finite part of $D - E$, and of class $C^1$ in the finite part of $D$ (Kellogg, pp. 150-172).

*Definition* 1. *The distribution will be termed admissible, if on $E$ the outer-normal derivative $f_n$ of the potential $f$ with respect to $D$ never vanishes. If moreover the potential $f$ has no degenerate critical point in the finite part of $D$, the distribution will be termed non-degenerate.**

* The restriction that $f$ be non-degenerate and admissible could be removed by the aid of the more general theorems on critical points in Morse II and in a recent paper by Morse, "The Critical Points of a Function of $n$ Variables," *Transactions of*

At great distances, we have the following developments in spherical harmonics for $f$ and its three partial derivatives of the first order $f_i (i = 1, 2, 3)$:

$$f(x) = m/r + h_1(x)/r^3 + \cdots,$$

$$f_i(x) = -mx_i/r^3 + h_2^i(x)/r^5 + \cdots, \tag{4.1}$$

where $m$ is the total mass of the distribution ($m = 0$ in the case of a double distribution), $r$ the distance from the origin to the variable point $(x)$, and $h$'s are the spherical harmonics of the orders indicated by their subscripts (Kellogg, pp. 143-145).

At the point at infinity, the potential $f$ and its partial derivatives will be assigned the value zero, their limiting value. They are regular at the point at infinity (Kellogg, p. 217), and harmonic in $D$ (Kellogg, p. 211). The values of $f_n$ on each boundary surface $E_k$ are of the same sign. We suppose the surfaces $E_i$ $(i = 1, 2, \cdots, p)$ are of positive type with respect to $f$ and $D$ (§ 2), and the surfaces $E_j$ $(j = p+1, p+2, \cdots, p+n)$ are of negative type, where $p + n = s$.

5. *Remarks on the critical points of f.* It is a well-known property of harmonic functions that the function $f$ has neither maximum nor minimum in $D - E$; in other words, $f$ has neither critical point of type 0 nor critical point of type 3 in $D - E$.

Suppose $h(x)$ is a harmonic function, but not a constant, in an open region $H$, finite or infinite. Let $H'$ be a closed regular subregion * of $H$. If $H'$ is finite, the critical points of $h$ in $H'$ constitute at most a finite number of isolated points, analytic curves, and analytic surfaces, † but can not fill any regular surface element, and the critical values of $h(x)$ in $H'$ are finite in num-

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*the American Mathematical Society*, Vol. 37 (1931), pp. 72-91. This would introduce a complexity which it seems desirable to avoid in a first treatment.

* A region is said to be regular, if it is a set of points in the extended 3-space and if its complete boundary consists of a finite number of closed regular surfaces.

† *A real analytic curve* is a closed, bounded, and connected set of points the coördinates $(x_1, x_2, x_3)$ of points neighboring a given point of which are representable as real analytic functions of a suitable real parameter $t$, will not all three functions identically constant. *A real analytic surface* is a closed, bounded, and connected set of points defined by setting one or more real analytic functions of $(x_1, x_2, x_3)$ equal to zero, such that the equations are not satisfied identically in the neighborhood of any point of the set and such that the points neighboring any point $P$ of the set are neither included on an analytic curve nor reduce to the point $P$.

ber (Kellogg, p. 262, pp. 276-277). If $H'$ is infinite, these statements can be proved as follows.*

In a spherical neighborhood $N$ of the point at infinity the function $h(x)$ and its partial derivatives of the first order $h_i(x)$, $(i = 1, 2, 3)$, are harmonic. Let us subject these four functions to an inversion in the unit sphere with center at the origin taken at a point not in $H'$:

$$x_i = x_i'/r'^2. \tag{5.1}$$

Let us designate the four transformed functions divided by $r'$ by $F(x')$ and $F_i(x')$ respectively. After the singularity at the origin is removed, each of these functions is harmonic in the transformed neighborhood $N'$ of the origin (Kellogg, pp. 232-233). These functions are analytic in $N'$, and vanish at the transformed origin. Hence, in $N'$, the locus of their common zeros consists of at most a finite number of isolated points, analytic curves, and analytic surfaces. If we note that the critical points of $h$ in $N$ are the common zeros of the four functions $F(x')$ and $F_i(x')$ in $N'$ and conversely, what is to be proved follows immediately.

The author has a proof for the fact that, of a set of points constituting a real analytic surface, there is always a subset of points constituting a regular surface element.† But the nonexistence of a regular surface element of critical points has been proved by Kellogg (Kellogg, p. 262). Hence we have the following theorem.

THEOREM 1. *If $h(x)$ is a function harmonic in an infinite open region of the extended space but not a constant, then in any closed regular subregion, the locus of the critical points of $h$ consists of at most a finite number of isolated points and analytic curves, and the number of critical values of $h$ is finite.*

COROLLARY. *In the region $D$ of § 4 the locus of the critical points of the non-degenerate admissible potential $f$ consists of at most a finite number of isolated points, and the number of critical values of $f$ is finite.*

6. *Nature of the locus at the point at infinity.* To study the locus $f(x) = 0$ in the neighborhood of the point at infinity, we invert it into a bounded locus. Suppose the origin is taken at a point at which there is no

* If a set of points in a regular infinite region is transformed by an inversion into a real analytic curve or surface, the set will be called a real analytic curve or surface.

† This will be published in a note elsewhere.

distribution and which is not on the locus $f = 0$. Then by the inversion (5.1) the potential $f$ is transformed into the function

$$g(x') \equiv r' \cdot F(x'), \tag{6.1}$$

where $F$, upon removing a singularity at the origin, becomes harmonic in a neighborhood of the origin (Kellogg, p. 232). From the expression for $f$ in (4.1), it is obvious that $F(0) = m$, where $m$ is the total mass of the distribution.

If $m$ is not zero, in a sufficiently small neighborhood of the origin $F$ will not vanish, and will have the same sign as $m$, and so will $g$.

If $m$ is zero, $F$ being harmonic, will be positive at some points and negative at some points in any neighborhood of the origin, and so will $g$. Thus we have the following results.

LEMMA 1. *Suppose $f$ is the potential due to an admissible distribution, the total mass of which is not zero. Then the point at infinity will be a point of proper minimum or maximum of $f$ according as the total mass is positive or negative.*

*If the total mass is zero, the point at infinity is not an isolated point on $f = 0$, and not a point of maximum or minimum of $f$.*

7. *Equipotential surfaces in the neighborhood of the point at infinity.* $m \neq 0$. We shall call a regular surface homeomorphic to an ordinary sphere a *sphere-like surface.*

We shall prove the following lemma.

LEMMA 2. *For a sufficiently small positive constant $e$ we have the following:*

(1) *If the total mass of the distribution is positive* [*negative*], *the equipotential locus $f = e^2$ [$f = -e^2$] consists in part at least of an analytic sphere-like surface $G[H]$ enclosing all the boundary surfaces $E_k (k = 1, 2, \cdots, s)$.*

(2) *The infinite closed region bounded by $G[H]$ is homeomorphic to a 3-cell.*

(3) *The function $f$ has no critical point in the finite part of the infinite closed region bounded by $G[H]$.*

(4) *At any point of $G[H]$ the outer normal derivative $f_n$ of $f$ with respect to the finite region bounded by $G[H]$ is negative* [*positive*].

*Proof.* In a neighborhood of the point at infinity the equipotential surfaces of the potential $f$ due to any admissible distribution are transformed by

the inversion (5.1) into the equipotential surfaces of $g(x') \equiv r' \cdot F(x')$ in the corresponding neighborhood $N$ of the origin. Subject $g(x')$ to the transformation

$$\bar{x} = F(x') \cdot x_i', \quad (i = 1, 2, 3). \tag{7.1}$$

The jacobian of this transformation at the origin is equal to $m^3$, which is not zero. The transformation is one-to-one and continuous in the neighborhood $N$, provided $N$ is sufficiently small.

Suppose $m$ is positive. Let $e$ be a sufficiently small positive constant and $\bar{r}$ the distance from the origin to $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. The locus of points satisfying $g(x') \leqq e^2$ in $N$ will be carried by (7.1) into the locus of points satisfying $\bar{r} \leqq e^2$, a spherical region. The function $\bar{r}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ has no critical point for which $0 < \bar{r} \leqq e^2$. Thus (1), (2), and (3) of the lemma are proved for $m$ positive. The proof is similar for $m$ negative.

Since $f$ has no critical point on $G[H]$ and since $f_n$ is normal to $G[H]$, $f_n$ can never vanish on $G[H]$. This proves (4) of the lemma.

The surface $G[H]$ of Lemma 2 will be called *the outer branch of* $f = e^2[f = -e^2]$.

8. *Critical points of* $f$. $m \neq 0$. Suppose $f$ is the potential due to a non-degenerate admissible distribution. There are two cases.

*Case* 1. $m < 0$. The outer branch $H$ of $f = -e^2$ of Lemma 2 together with boundary surfaces $E_1, E_2, \cdots, E_{p+n}$ of $D$ (§ 4) bounds a closed bounded region which we take as the region $R$ of § 3. The boundary of $R$ we have designated by $B$. The set $B'$ of boundary surfaces of positive type with respect to $f$ and $R$ (§ 3) consists of $p + 1$ surfaces, $H, E_1, E_2, \cdots, E_p$, and the set $B''$ of boundary surfaces of negative type with respect to $f$ and $R$ consists of the remaining $n$ surfaces, $E_{p+1}, E_{p+2}, \cdots, E_{p+n}$. In the region $R$, the function $f$ is admissible under the new boundary conditions of § 3.

Let the first Betti number of $E_k$ be $2g_k$. Then the first Betti number of either one of the two regions into which the extended 3-space (§ 4) is divided by $E_k$ is $g_k$ (Morse II, Corollary to Theorem 9 and the second part of Theorem 11, pp. 816-817). The Betti numbers of the set $B''$ of boundary surfaces are the following:

$$R'_0 = n, \; R'_1 = 2 \sum_{j=p+1}^{p+n} g_j, \; R'_2 = n, \; R'_3 = 0.$$

Since no two of the surfaces $E_k$ enclose a common point, the Betti numbers of the region complimentary to $R$ with respect to the extended 3-space are therefore

$$p+n+1, \sum_{k=1}^{p+n} g_k, 0, 0.$$

From another theorem in analysis situs (Morse II, Corollary to Theorem 11, p. 817), the Betti numbers of $R$ are the following:

$$R_0 = 1, R_1 = \sum_{k=1}^{p+n} g_k, R_2 = p+n, R_3 = 0.$$

The function $f$ is harmonic in $R$ and therefore in the interior of $R$ it has neither critical point of type 0 nor critical point of type 3, that is, it has $M_0^+$ and $M_3^-$ both equal to zero. From Theorem $B$ and (2.2), we have for $f$ in $R$

$$M_1^- = n-1, \qquad M_1^+ = \sum_{i=1}^{p} g_i + b,$$
$$M_2^- = \sum_{j=p+1}^{p+n} g_j + b, \quad M_2^+ = p,$$

where $b$ is an integer, positive, negative, or zero, but not less than the larger one of the two numbers:

$$-\sum_{i=1}^{p} g_i, \quad -\sum_{j=p+1}^{p+n} g_j.$$

From Theorem B and (2.1) we have for $f$ in $R$

$$M_1 = n-1+\sum_{i=1}^{p} g_i + b,$$
$$M_2 = p \qquad + \sum_{j=p+1}^{p+n} g_j + b. \tag{8.1}$$

*Case* 2. $m > 0$. The outer branch $G$ of $f = e^2$ of Lemma 2 together with the boundary surfaces $E_1, E_2, \cdots, E_{p+n}$ bounds a closed bounded region which we take as the region $R$. The boundary of $R$ we have designated by $B$. The set $B'$ of boundary surfaces of positive type with respect to $f$ and $R$ consists of $p$ surfaces, $E_1, E_2, \cdots, E_p$, and the set $B''$ of boundary surfaces of negative type with respect to $f$ and $R$ consists of the remaining $n+1$ surfaces, $G, E_{p+1}, E_{p+2}, \cdots, E_{p+n}$. In the region $R$ the function $f$ is admissible under the new boundary conditions of § 3. As in the preceding case, we find for $f$ in $R$

$$M_1^- = n, \quad M_1^+ = \sum_{i=1}^{p} g_i + b,$$
$$M_2^- = \sum_{j=p+1}^{p+n} g_j + b, \quad M_2^+ = p-1;$$

and

$$M_1 = n + \sum_{i=1}^{p} g_i + b,$$
$$M_2 = p-1+\sum_{j=p+1}^{p+n} g_j + b; \tag{8.2}$$

where $b$ has the same meaning as in the preceding case. Hence we have the following theorem.

THEOREM 2. *Suppose there is a non-degenerate admissible distribution over, or in the closed finite regions bounded by, $p+n$ distinct closed regular surfaces, $E_1, E_2, \cdots, E_{p+n}$, of class $C^3$, no two of these surfaces enclosing a common point. Suppose the first Betti number of $E_k (k = 1, 2, \cdots, p+n)$ is $2g_k$. Suppose $E_i (i = 1, 2, \cdots, p)$ is of positive type with respect to $f$ and the space exterior to it, and $E_j (j = p+1, p+2, \cdots, p+n)$ is of negative type with respect to $f$ and the space exterior to it* (§ 2).

*Then, if the total mass of the distribution is not equal to zero, in the space exterior to all these $p+n$ surfaces the potential $f$ due to this distribution has at least*

$$p+n-1+\left|\sum_{i=1}^{p} g_i - \sum_{j=p+1}^{p+n} g_j\right|$$

*critical points. If $f$ has more than this number of critical points, the excess is even.*

*According as the total mass is negative or positive, the respective numbers $M_1$ and $M_2$ of critical points of type 1 and type 2 are given by* (8.1) *or* (8.2).

9. *Equipotential surfaces of a redefined function $g$ in the neighborhood of the infinite branch of $f = 0$. $m = 0$.* The function $f$ for $m = 0$ in the region $D$ of § 4 fulfills the conditions for an admissible function in $D$ under the new boundary conditions of § 3 except that $D$ is not bounded. But the argument for redefinition of a function in $R$ in a neighborhood of $B''$ as described in § 3 can still be applied to redefinition of $f$ in $D$ in a neighborhood of the boundary $E$ of $D$. Let the new function obtained from such a redefinition of $f$ in $D$ in a neighborhood of $E$ be $g$. We shall still designate the new region by $D$ and its boundary by $E$. The function $g$ in the new region $D$ fulfills the conditions for an admissible function in $D$ in the sense of § 2 except that $D$ is not bounded.

By an inversion with the center of inversion not on the locus $g = 0$, the locus $g = 0$ will be transformed into a bounded locus through the origin. We shall say that a point of $g = 0$ is *connected to the point at infinity* if it is transformed by the inversion into a point which can be connected to the origin on the transformed locus. We shall call the locus of points of $g = 0$ which are connected to the point at infinity the *infinite branch* of $g = 0$. Similarly the infinite branch of $f = 0$ will be defined.

Let us suppose $f$ is non-degenerate. From the Corollary to Theorem 1, in $D$ the number of critical values of $f$, and therefore of $g$, is finite. In

redefinition we can, and shall, assume that the boundary values $c'$ and $c''$ (§ 2) of $g$ are both different from zero. It is then possible to take a positive constant $e$ so small that no critical value of $g$ lies between 0 and $+e^2$ or between $-e^2$ and 0, that $c'' < -e^2 < +e^2 < c'$, and that $\pm e^2$ are not critical values of $g$. The open set of points satisfying $-e^2 < g < e^2$ contains the point at infinity. Let us suppose the above center of inversion is not in this set. Then this set will be transformed into an open bounded set which consists of one or more continua. As a convention we shall say that the subset of points satisfying $-e^2 < g < e^2$, which is transformed into the continuum containing the origin, is a continuum, and we shall call it the *infinite continuum*. In the infinite continuum the infinite branch of $g = 0$ is contained but not the boundary $E$. The boundary of the infinite continuum consists of $u \geqq 1$ closed regular surfaces $S_\alpha$ $(\alpha = 1, 2, \cdots, u)$ of class $C^2$ satisfying $g = -e^2$, and $v \geqq 1$ closed regular surfaces $S_\beta$ $(\beta = u+1, u+2, \cdots, u+v)$ of class $C^2$ satisfying $g = e^2$. $S_\alpha$ will be called the $u$ outer branches of $g = -e^2$ and $S_\beta$ the $v$ outer branches of $g = e^2$. Each of these $u+v$ surfaces is a 2-manifold. No two of these surfaces enclose a common point. The sum of the finite closed regions bounded by these surfaces and the infinite continuum is the extended 3-space. It follows from a method of proof given by Morse (Morse I, pp. 356-360) that the Betti Numbers of each of these finite regions are independent of the constant $e$, for $e$ sufficiently small and positive.

Since the value of $g$ is $-e^2$ on $S_\alpha$ and $e^2$ on $S_\beta$, and since $\pm e^2$ are not critical values, on each of the surfaces $S_\alpha$ and $S_\beta$, the outer normal derivative of $g$ with respect to the finite space it bounds never vanishes. It is then easy to see that at each point of $S_\alpha$ the outer normal derivative of $g$ is positive, but at each point of $S_\beta$ the outer normal derivative of $g$ is negative. Moreover any critical point of $g$ not on the infinite branch of $g = 0$ is enclosed by some one of the surfaces $S_\alpha$ and $S_\beta$.

These results will be summed up in the following lemma.

LEMMA 3. *Suppose $f$ is the potential due to a non-degenerate admissible distribution, the total mass of which is zero. Suppose $g$ is the function obtained by redefining $f$ in a neighborhood of $E$ (in the way of redefining $f$ in a neighborhood of $B''$ as described in § 3). For a sufficiently small positive constant $e$, there are $u \geqq 1$ closed regular surfaces $S_\alpha$ $(\alpha = 1, 2, \cdots, u)$ of class $C^2$ satisfying $g = -e^2$, the $u$ outer branches of $g = -e^2$, and $v \geqq 1$ closed regular surfaces $S_\beta$ $(\beta = u+1, u+2, \cdots, u+v)$ of class $C^2$ satisfying $g = e^2$, the $v$ outer branches of $g = e^2$, with the following properties:*

(1) *Each critical point of $f$, not a critical point on the infinite branch*

*of $f=0$ is enclosed by some one of the $u+v$ surfaces $S_h$ ($h=1, 2, \cdots, u+v$).*

(2) *Each of the surfaces, $E_1, E_2, \cdots, E_{p+n}$, is enclosed by some one of the surfaces $S_h$.*

(3) *No two of the $u+v$ surfaces $S_h$ enclose a common point.*

(4) *At each point of $S_\alpha$ the outer normal derivative of $g$ with respect to the finite space it bounds is positive, and at each point of $S_\beta$ the outer normal derivative of $g$ with respect to the finite space it bounds is negative.*

10. *Critical points of $f$. $m=0$.* Suppose the boundary surfaces, $E_1, E_2, \cdots, E_{p+n}$, of the region $D$ (§ 4) of positive and negative types with respect to $g$ and $D$, which are enclosed by the surface $S_h$ ($h=1, 2, \cdots, u+v$) are respectively the following $p_h$ surfaces:

$$E_{h,1},\ E_{h,2}, \cdots, E_{h,p_h};$$

and the following $n_h$ surfaces:

$$E_{h,p_h+1},\ E_{h,p_h+2}, \cdots, E_{h,p_h+n_h}.$$

Then we have obviously

$$\sum_{h=1}^{u+v} (p_h + n_h) = s.$$

Suppose the first Betti number of $S_h$ is $2q_h$. Let $D_h$ designate the finite closed region bounded by $S_h$ and the $p_h + n_h$ surfaces enclosed by $S_h$. Obviously a boundary surface of $D$ of positive or negative type with respect to $g$ and $D$, which is a boundary surface of $D_h$, is of the same type with respect to $g$ and $D_h$.

Now, from (2), (3), (4) of Lemma 3, the function $g$ is an admissible function in $D_h$ under the new boundary conditions of § 3. By the method used in Case 1 of § 8, we find that in $D_\alpha$ ($\alpha=1, 2, \cdots, u$) the respective numbers of critical points of type 1 and type 2 of $g$ are

$$M_{1,\alpha} = n_\alpha - 1 + \sum_{\rho=1}^{p_\alpha} g_{\alpha,\rho} + q_\alpha + b_\alpha,$$

$$M_{2,\alpha} = p_\alpha + \sum_{\sigma=p_\alpha+1}^{p_\alpha+n_\alpha} g_{\alpha,\sigma} + b_\alpha, \tag{10.1}$$

where $g_{ij}$ is the first Betti number of $E_{ij}$, and where $b_\alpha$ is an integer, positive, negative, or zero, but not less than the larger one of the two numbers:

$$-\sum_{\rho=1}^{p_\alpha} g_{\alpha,\rho} - q_\alpha, \quad -\sum_{\sigma=p_\alpha+1}^{p_\alpha+n_\alpha} g_{\alpha,\sigma}.$$

Similarly, by the method used in Case 2 of § 8, we find that in $D_\beta$

($\beta = u+1, u+2, \cdots, u+v$) the respective numbers of critical points of type 1 and type 2 of $g$ are

$$M_{1,\beta} = n_\beta + \sum_{\rho=1}^{p_\beta} g_{\beta,\rho} + b_\beta,$$

$$M_{2,\beta} = p_\beta - 1 + \sum_{\sigma=p_{\beta^+1}}^{p_{\beta^+n_\beta}} g_{\beta,\sigma} + q_\beta + b_\beta, \tag{10.2}$$

where $b_\beta$ is an integer, positive, negative, or zero, but not less than the larger one of the two numbers

$$-\sum_{\rho=1}^{p_\beta} g_{\beta,\rho}, \quad -\sum_{\sigma=p_{\beta^+1}}^{p_{\beta^+n_\beta}} g_{\beta,\sigma} - q_\beta.$$

Summing up (10.1) for $\alpha = 1, 2, \cdots, u$, and (10.2) for $\beta = u+1, u+2, \cdots, u+v$, we find that the respective numbers of critical points of type 1 and type 2 of $g$ in all the regions $D_h$ ($h = 1, 2, \cdots, u+v$) are

$$M_1 = n - u + \sum_{i=1}^{p} g_i + \sum_{\alpha=1}^{u} q_\alpha + b,$$

$$M_2 = p - v + \sum_{j=p+1}^{p+n} g_j + \sum_{\beta=u+1}^{u+v} q_\beta + b, \tag{10.3}$$

where $b$ is an integer, positive, negative, or zero, but not less the larger one of the two numbers:

$$-\sum_{i=1}^{p} g_i - \sum_{\alpha=1}^{u} q_\alpha, \quad -\sum_{j=p+1}^{p+n} g_j - \sum_{\beta=u+1}^{u+v} q_\beta.$$

From (1) of Lemma 3, we have then the following theorem.

THEOREM 3. *Suppose the hypotheses of Theorem 2 hold, and the total mass of the distribution is equal to zero. Suppose the first Betti numbers of the $u$ outer branches $S_\alpha$ ($\alpha = 1, 2, \cdots, u$) of $g = -e^2$ and the $v$ outer branches $S_\beta$ ($\beta = u+1, u+2, \cdots, u+v$) of $g = e^2$ (Lemma 3) are $2q_\alpha$ and $2q_\beta$ respectively.*

*Then in the space exterior to the surfaces $E_k$ ($k = 1, 2, \cdots, p+n$) the potential $f$ has at least*

$$p + n - (u+v) + \left| \sum_{i=1}^{p} g_i - \sum_{j=p+1}^{p+n} g_j + \sum_{\alpha=1}^{u} q_\alpha - \sum_{\beta=u+1}^{u+v} q_\beta \right|$$

*critical points not on the infinite branch of $f = 0$.*

*The respective numbers $M_1$ and $M_2$ of these critical points of type 1 and type 2 are given by (10.3).*

It is interesting to note that, so far as formulas go, (8.1) and (8.2) are particular cases of (10.3).

11. *Static charges on conductors in equilibrium.* The case of static charges on conductors (Kellogg, p. 176) in equilibrium is a special case of simple distribution of matter over closed surfaces. Suppose the surfaces $E_k$, $(k = 1, 2, \cdots, p+n)$, of § 4 are the surfaces of $p+n$ finite conductors, and the surface density of the static charges at a point $P$ of any one of the surfaces is $m(P)$. Let $f_n(P)$ designate the outer normal derivative at $P$ of the potential $f$ due to this distribution with respect to the conductor on which $P$ lies. It is well-known that when the distribution is in equilibrium,

$$f_n(P) = 4\pi m(P)$$

(Kellogg, p. 164). Hence *an admissible distribution of static charges in equilibrium* over a finite number of finite conductors, the surfaces of which are regular surfaces of class $C^3$, may de defined as a distribution in equilibrium, the surface density of which is continuous on each conductor and never vanishes. Hence we have the following theorem at once.

THEOREM 4. *The preceding two theorems hold for the potential due to a non-degenerate distribution of static charges in equilibrium over the $p+n$ finite conductors, the surfaces of which are the surfaces, $E_1, E_2, \cdots, E_{p+n}$, of those theorems.*

## III. THE NON-DEGENERATE NEWTONIAN POTENTIAL DUE TO A FINITE NUMBER OF POINT MASSES.

12. *The potential. The equipotential surfaces in the neighborhoods of the point masses.* Suppose in the extended 3-space there are $p$ masses $m_i > 0$ $(i = 1, 2, \cdots, p)$ at the $p$ finite distinct points $P_i = (x^i)$, and $n$ masses $m_j < 0$ $(j = p+1, p+2, \cdots, p+n)$ at the $n$ finite points $P_j = (x^j)$, distinct among themselves and from the $p$ points $P_i$. Then the Newtonian potential due to these point masses is the following function:

$$F(x) = m_k/r_k, \quad (k = 1, 2, \cdots, p+n),$$

where $r_k$ is the distance from the point $P_k$ to the variable point $(x)$. The function $F$ becomes positively infinite at $P_i$ and negatively infinite at $p_j$. The distribution will be said to be *non-degenerate* if $F$ has no degenerate critical point in the finite part of the 3-space.

We shall prove the following lemma.

LEMMA 4. *Suppose $F$ is the potential due to a distribution of $p$ positive point masses at $P_i$ $(i = 1, 2, \cdots, p)$ and $n$ negative point masses at $P_j$ $(j = p+1, p+2, \cdots, p+n)$.*

(1) *For a sufficiently large positive constant $L$, the equipotential locus*

*$F(x) = L$ consists of $p$ analytic sphere-like surfaces $E_i$, $E_i$ enclosing $P_i$; and the equipotential locus $F(x) = -L$ consists of $n$ analytic sphere-like surfaces $E_j$, $E_j$ enclosing $P_j$.*

(2) *The finite closed region bounded by each of these sphere-like surfaces is a 3-cell.*

(3) *The function $F$ has no critical point in the finite closed region bounded by any one of these sphere-like surfaces.*

(4) *At each point of $E_i$ $[E_j]$, the outer normal derivative of $F$ with respect to the infinite region bounded by $E_i$ $[E_j]$ is positive [negative].*

*Proof.* Let us confine our attention on a neighborhood of one of the points $P_i$. Without loss of generality, we may take one such point at the origin. Then, in a neighborhood of the origin, the potential can be represented in the from:

$$F(x) = u \cdot [1/r - w(x)]$$

where $u$ is a positive constant, the quantity of mass at the origin, $r$ the distance from the origin to the point $(x)$, and $w(x)$ a function analytic in that neighborhood.

Let us introduce the transformation

$$x'_i = x_i/[1 - rw(x)], \quad (i = 1, 2, 3). \tag{12.1}$$

The transformation is continuous at the origin, and analytic at any point in a sufficiently small neighborhood of the origin except possibly at the origin. It can be easily shown that the transformations is of class $C^1$ at the origin, and has a non-vanishing jacobian. Hence the transformation is continuous and one-to-one in a neighborhood $N$ of the origin.

If the positive constant $L$ is taken sufficiently large, the set of points satisfying the inequality $F \geqq L$ has a subset $S$ of points in $N$ and has no point on the boundary of $N$. Moreover, by virtue of (12.1), we have

$$u \cdot (1/r - w(x)) \equiv u/r',$$

where $r'$ is the distance from the transformed origin to $(x')$. The set $S$ is therefore carried by the transformation (12.1) into the set of points satisfying

$$r' \leqq u/L,$$

which constitutes a 3-cell. Therefore the set $S$ constitutes a 3-cell. The boundary of the set $S$ is, of course, an analytic surface.

It is easy to see that the locus $F(x) = L$ consists of exactly $p$ analytic sphere-like surfaces, one enclosing each positive point mass. Similar statements can be made for $F(x) = -L$.

Thus, the statements (1), (2), and (3) of the lemma follow from the transformation (12.1). Since $F$ has no critical point on these $p+n$ sphere-like surfaces, the normal derivative of $F$ can never vanish on them. Statement (4) then follows at once.

13. *Critical points of F.* The function $F$ and its partial derivatives of the first order are regular at infinity. Thus the Corollary to Theorem 1 holds for $F$. In fact, the discussions in § 6 and § 7 hold for $F$ almost word for word. So does the discussion in § 9, if we omit the redefinition there because it is unnecessary for $F$, and if we regard $g$ there as $F$ itself. Hence Lemmas 1-3 with obvious modifications hold for $F$. On setting $g_k$ $(k=1, 2, \cdots, p+n)$ in Theorem 2 equal to zero, we obtain the following theorem.

THEOREM 5. *Suppose there is a non-degenerate distribution of $p$ positive and $n$ negative point masses, the total mass of which is not zero. Then in the finite space the potential $F$ due to this distribution has at least $p+n-1$ critical points. If $F$ has more than this number of critical points, the excess is even.*

*If the total mass is negative [positive], $F$ has at least $n-1$ $[n]$ critical points of type one and $p$ $[p-1]$ critical points of type two. The excess of critical points of type one, if any, is equal to that of type two.*

Similarly, from the proof of Theorem 3 and from the nature of the constants $b$'s there, we have the following theorem.

THEOREM 6. *Suppose there is a non-degenerate distribution of $p$ positive and $n$ negative point masses, the total mass of which is zero. Suppose the first Betti numbers of the $u$ outer branches $S_\alpha$ $(\alpha=1, 2, \cdots, u)$ of $F=-e^2$ and the $v$ outer branches $S_\beta$ $(\beta=u+1, u+2, \cdots, u+v)$ of $F=e^2$ are $2q_\alpha$ and $2q_\beta$ respectively* (cf. Lemma 3).

*Then in the finite space the potential $F$ due to this distribution has at least*

$$p+n-(u+v)+\sum_{k=1}^{u+v} q_k$$

*critical points not on the infinite branch of $F=0$. If $F$ has more than this number of critical points not on the infinite branch of $F=0$, the excess is even.*

*The respective numbers of these critical points of type 1 and type 2 are*

$$M_1=n-u+\sum_{\alpha=1}^{u} q_\alpha+b, \quad M_2=p-v+\sum_{\beta=u+1}^{u+v} q_\beta+b,$$

*where $b$ is a positive integer or zero.*

# COMPLETE SETS OF CONJUGATES UNDER A GROUP.

By G. A. MILLER.

1. *Theorems relating to complete sets of conjugate operators.* If a group $G$ is not the identity it cannot involve a complete set of conjugate operators which includes more than one-half of its operators since each operator is transformed into itself by all of its powers. Moreover, when $G$ involves a complete set of conjugate operators which is composed of one-half of its operators then each of them must be of order 2. The product of one of them and any other must be of odd order since this product is transformed into its inverse by each of these two operators of order 2 which are transformed into themselves under $G$ only by their own powers. Hence exactly one-half of the operators of $G$, including the identity, are of odd order.

The product of two such operators of odd order could not be of order 2 since the products obtained by multiplying one of them successively into all of these operators of order 2 give all of these operators in some order. It therefore results that these operators of odd order constitute a subgroup of index 2 under $G$. This subgroup must be abelian since each of its operators corresponds to its inverse in an automorphism of $G$. Moreover, every abelian group of odd order can be extended by an operator of order 2 which transforms each of its operators into its inverse and the resulting group will be such a $G$. That is, *a necessary and sufficient condition that a group which is not the identity contains a single set of conjugate operators composed of as many as one-half of the operators of the group is that it can be obtained by extending an abelian group of odd order by an operator of order 2 which transforms into its inverse every operator of this abelian group.* This extended group must therefore be either a dihedral group or an extended dihedral group involving a subgroup of odd order and of index 2.

Before determining properties of the groups which involve two complete sets of conjugates such that each set is composed of exactly one-third of the operators of the group it is desirable to establish a theorem which will be very useful in this determination. Suppose that $s$ is an operator of order 3 and that the following equation is satisfied:

$$s^2tst = ts^2ts.$$

From these conditions it follows directly that the three conjugates of $t$ under the powers of $s$ are commutative. It is also clear that the given equation is satisfied whenever each of the two operators $st$ and $s^2t$ is of order 3. Hence

the following theorem: *If $s$ is an operator of order 3 and if $t$ is any operator such that $s^2tst = ts^2ts$ then the three conjugates of $t$ under the powers of $s$ are relatively commutative.*

If $G$ involves two complete sets of conjugate operators such that each of them is composed of one-third of the operators of $G$ then we can select a set of $m$ of these operators, where $m$ is one-third of the order of $G$, which have the property that the products of a given one of them $s_1$ into all of them are all of order 3. We shall denote these sets of operators of order 3 by

$$s_1,\ s_2,\ \cdots,\ s_m \qquad s_1^2,\ s_1s_2,\ \cdots,\ s_1s_m$$

respectively. From the theorem noted above it results that these sets have no common operator since it may be assumed without loss of generality that each of these operators is transformed into itself only by its own powers. It also results from this theorem that the order of each of the $m$ distinct operators $1,\ s_1^2s_2,\ \cdots,\ s_1^2s_m$ is not equal to 3. This set of $m$ operators is therefore composed of all the operators of $G$ whose orders are not equal to 3 and the same set would result if we would start with any one of the other operators of order 3 instead of starting with $s_1$.

Since each of the products obtained by multiplying these $m$ operators successively by one of the remaining operators of $G$, or by its square, is of order 3, it results that the product of any two of these $m$ operators, or the square of any one of them, is not of order 3. Hence these $m$ operators constitute an invariant subgroup $H$ of index 3 under $G$ whose order is prime to 3. If $H$ involved more than one Sylow subgroup of a given order it would involve two non-commutative operators which would be conjugate under an operator of order 3 of $G$. This is impossible according to the theorem noted above. Hence we have proved the following theorem: *If a group involves two complete sets of conjugates such that each set is composed of exactly one-third of the operators of the group then it contains an invariant subgroup of index 3 which is the direct product of its Sylow subgroups and the order of each of these subgroups is of the form $3k + 1$.*

If a group $G$ of order $g$ involves a complete set of $g/n$, $n < g$, conjugate operators then the order of its central must divide $n$ and be less than $n$. In particular, when $n$ is a prime number the central of $G$ is the identity. In this case the order of $G$ cannot be divisible by the square of this prime number since every subgroup of order $p^m$, $p$ being a prime number, contained in $G$ is found in at least one of the Sylow subgroups of $G$ whose order is a power of $p$. These statements are illustrated by the theorems noted above and may be employed in giving alternate proofs of these theorems. In particular, it

follows directly therefrom that the order of a group involving a set of conjugates composed of exactly one-half of its operators is divisible by 2 but not by 4 and hence it contains a subgroup of index 2 composed of its operators of odd order including the identity.

2. *Groups involving a small number of complete sets of conjugate non-invariant operators.* There is no group which involves one and only one complete set of conjugate non-invariant operators since such a set could not involve more than one-half of the operators of the group and the central quotient group of a non-abelian group cannot be cyclic. If a group involves just two such sets one of them must involve one-half of the operators of the group since the central could not involve more than one-fourth of these operators. Hence the order of the group is 6 and it is the non-cyclic group of this order. It results directly from the theorem noted above that the number of different complete sets of conjugate non-invariant operators in a group involving a set of conjugates composed of half its operators is $g/4 + \frac{1}{2}$. When the number of these sets of conjugates is a given number $h > 1$ the number of these groups is equal to the number of the abelian groups of order $2h - 1$. In particular, there is at least one such group for every value of $h > 1$.

There is no upper limit for the possible orders of the groups which involve no set of conjugate non-invariant operators since every abelian group has this property. On the contrary, when the number of these sets of conjugates is any given number $k > 1$, the order of the possible groups has an upper limit and hence the number of these groups is also limited. To prove this fact it may be noted that if $g_1, g_2, \cdots, g_k$ represent the orders of the subgroups which transform into itself one of each set of such conjugate operators then the order of the central of $G$ must divide the highest common factor of these $g$'s and it must be less than the smallest of them. In particular, when at least one of these $g$'s is a prime number the central of $G$ must be the identity. As the sum of the reciprocals of these $g$'s is at least equal to $\frac{3}{4}$ at least one of them cannot exceed $4k/3$ and the order of the central can therefore not exceed $2k/3$. After this smallest $g$ has been selected we can proceed similarly with the remaining $k - 1$ $g$'s. Hence the following theorem: *Only a finite number of groups contain any given number of complete sets of conjugate non-invariant operators.*

If a group contains three complete sets of conjugate non-invariant operators and one of its $g$'s is 2 it must be the dihedral group of order 10 according to the theorem noted above. If its smallest $g$ is 3 it is the tetrahedral group, and if its smallest $g$ is 4 it must be a non-abelian group of

order 8. There are therefore four and only four groups which have the property that each of them involves exactly three complete sets of conjugate non-invariant operators. These are the two non-abelian groups of order 8, the tetrahedral group, and the dehedral group of order 10.

If a group contains four complete sets of conjugate non-invariant operators and one of its $g$'s is 2, it must be the dihedral group of order 14. When two of its $g$'s are 3, it is the semi-metacyclic group of order 21, and when only one of them is 3 and another is 4, it may be either the octahedral group or the icosahedral group. It remains to consider the cases when the smallest $g$ exceeds 3. If three of these $g$'s are 4 then $G$ is the holomorph of the group of order 5. When two of them are 4 and the others exceed 4 then $G$ is either the dihedral or the dicyclic group of order 12. It is easy to verify that these are the only possible cases which correspond to a possible group and hence there are exactly seven groups which separately have the property that there appear therein four and only four complete sets of conjugate non-invariant operators.

3. *Groups involving only one complete set of conjugate non-invariant subgroups.* Let $G$ represent any group which involves one and only one complete set of conjugate non-invariant subgroups. The order of these cyclic subgroups must be of the form $p^m$, $p$ being a prime number. Moreover, there cannot be more than one Sylow subgroup of order $q^n$, $q$ being different from $p$, in $G$ and every subgroup of this Sylow subgroup must be invariant thereunder and hence the Sylow subgroup must be abelian when $q > 2$, and either abelian or Hamiltonian when $q = 2$. The subgroup of index $p$ contained in a non-invariant cyclic subgroup of order $p^m$ must be invariant under $G$ and hence a characteristic subgroup of $G$ because it appears in each of its non-invariant subgroups. This subgroup and the Sylow subgroups of $G$ whose orders are prime to $p$ must generate a characteristic subgroup of $G$, which is the direct product of its Sylow subgroup and involves no invariant operator of $G$ except the identity and possibly operators whose orders are powers of $p$.

From the preceding paragraph it results directly that the order of $G$ may be divisible by more than two prime numbers and when it is divisible by two such numbers it is of the form $p^m q^n$, $q$ being a prime of the form $1 + kp$. Moreover, it is obvious that when $q$ is any prime number of this form it is possible to construct one and only one group of order $pq^m$, where $m$ is an arbitrary natural number, which involves one and only one complete set of conjugate non-invariant subgroups of order $p^m$. The number of these subgroups is $q^n$. This proves the following theorem: *A necessary and sufficient condition that a non-abelian group $G$ whose order involves only two*

*prime numbers contains one and only one complete set of non-invariant conjugate subgroups is that its order is of the form $p^m q^n$, $p$ and $q$ being prime numbers and $q$ of the form $1 + kp$, and that $G$ contains an invariant abelian subgroup of index $p$ and also an operator of order $p^m$.*

It remains to consider the case when the order of $G$ is a power of $p$. The conjugate cyclic subgroups of order $p^m$ in $G$ must now generate an invariant subgroup $H$ of $G$ while in the preceding case they generate $G$ itself. The former fact results directly from the theorem that in the group generated by a complete set of conjugate subgroups of a prime power group, these subgroups cannot all be conjugate while in the group generated by the Sylow subgroups of a group they are always all conjugate. The quotient group $G/H$ is cyclic. When $p > 2$ this follows directly from the theorem that if each of two non-commutative operators whose orders are powers of $p$ transforms the group generated by the other into itself then the product of a power of one of these operators into the other generates a group which is not invariant under the other. This theorem applies also to the case when $p = 2$ and at least one of these two operators has an order which exceeds 4. If an operator of $G$ which is not commutative with the subgroup of order $2^m$ generated by an operator $t$ were of order 4 then $t$ could not be either of order 2 or of order 4. It therefore results that $G/H$ is also cyclic when $p = 2$.

The cyclic subgroup of order $p^{m-1}$ which is common to all the conjugate subgroups of order $p^m$ contained in $G$ must be in the central of $G$ since the operators of $G$ which correspond to a generator of $G/H$ must be commutative with all of these operators. The conjugate subgroups of order $p^m$ must therefore correspond to an invariant subgroup of order $p$ in the quotient group corresponding to the group generated by their common subgroup of order $p^{m-1}$. Hence these conjugate subgroups generate an abelian group and the commutator subgroup of $G$ is of order $p$. Hence $m$ cannot exceed 1 and $H$ must be the non-cyclic group of order $p^2$. This proves the following theorem: *If a group of order $p^m$, $p$ being a prime number, has the property that it contains one and only one complete set of non-invariant conjugate subgroups then it is the non-abelian group which is conformal to the abelian group of this order and of type $(m-1, 1)$.*

It should be noted that each of the two systems of groups which involve together all the groups containing one and only one complete set of non-invariant conjugate subgroups is composed of an infinite number of distinct groups. On the contrary, it was proved above that if a group contains a finite number $k$ of complete sets of non-invariant conjugate operators then its order has an upper limit which is a function of $k$, and hence the number

of such groups for a given value of $k$ is finite. It may also be noted that it follows directly from the method used above in the study of the groups of order $p^m q$ that there are groups of order $p^m q^n$ which involve exactly $n$ complete sets of non-invariant conjugate subgroups, where $n$ is an arbitrary natural number. To construct such a group it is only necessary to assume that the Sylow subgroup of order $q^n$ is cyclic. This proves the existence of an infinite number of groups which involve separately exactly a given arbitrary number of complete sets of non-invariant conjugate subgroups.

4. *Complete sets of conjugate subgroups of a prime power group.* If in any permutation group of degree $n$ the subgroup composed of all the permutations which omit one letter is of degree $n - k$, $k > 1$, then every permutation of this group which omits one letters omits a multiple of $k$ letters. In other words, its systems of imprimitivity composed of $k$ letters each have the property that no permutation of the group permutes some but not all of the letters of a system, and vice versa. If the letters of such a permutation group correspond to a complete set of conjugate non-invariant subgroups or non-invariant operators of a group of order $p^m$ then $k = p^\alpha$, $\alpha > 0$. Hence such a complete set of conjugates always involves $p^\alpha$ which have the property that each of them is transformed into itself by every other one of these $p^\alpha$ conjugates. Every operator of the group which transforms one of these $p^\alpha$ conjugates into itself must transform a multiple of $p^\alpha$ of them separately into themselves.

When a group $G$ of order $2^m$ contains a complete set of $2^{m-2}$ conjugate subgroups of order 2 the generator $s_1$ of one of these subgroups must be transformed into itself by the generator $s_2$ of another and $s_1$, $s_2$ generate the four-group. The subgroup of order 8 which contains this four-group must be the octic group since $s_1$ and $s_2$ are conjugate thereunder. The subgroup of order 16 which contains this octic group must involve an operator which transforms $s_1$ into itself multiplied by an operator of order 4 and is commutative with this operator. This transforming operator must therefore be of order 8 and the subgroup of order 16 must be formed by extending the cyclic group of order 8 by an operator of order 2 which transformed its operators either into their seventh or into their third powers. When the order of $G$ exceeds 16 it could not be the latter and the subgroup of order 2 which contains the former, transforms the non-invariant operators of order 2 contained therein into themselves multiplied by an operator of order 8. Hence this group of order 32 contains an operator of order 16. As this process can be continued until $G$ is reached we have established the following theorem: *If a group of order $2^m$ contains a complete set of $2^{m-2}$ conjugate subgroups*

*of order 2, it is either the dihedral group of order $2^m$ or the group obtained by extending the cyclic group of order $2^{m-1}$ by an operator of order 2 which transforms each of its operators into its $2^{m-2}-1$ power.* In the former case $m > 2$ there are two such sets of conjugates while in the latter case $m > 3$ there is only one such set. When $m = 3$ the octic group is the only one which satisfies this condition while there are always two such groups when $m > 3$.

When a group $G$ of order $p^m$ contains a complete set of $p^{m-2}$ conjugate subgroups of order $p > 2$ the generator $s_1$ of one of these subgroups must again be transformed into itself by the generator $s_2$ of another and $s_1$, $s_2$ generate the non-cyclic group of order $p^2$. The subgroup of order $p^3$ which involves this non-cyclic subgroup must be the non-abelian group of this order which involves no operator of order $p^2$ whenever $m > 3$ since $s_1$ is transformed into itself by only $p^2$ of the operators of $G$. This is also true of the subgroup of order $p^4$ which contains this subgroup of order $p^3$ when $p > 3$ and $m > 4$, but when $p = 3$ this subgroup of order $p^3$ involves the abelian group of order $p^3$ and of type $(2, 1)$ while each of its remaining operators is of order 3. It may be noted that when $G$ contains an abelian subgroup of index $p$ then the number of the independent generators of this subgroup cannot exceed $p$ for if this number were larger than $p$ this subgroup would involve a characteristic subgroup of type $(1, 1, 1, \cdots)$ containing more than $p$ invariants and only one subgroup of order $p$ which would be invariant under $G$. As this is obviously impossible it results that *when a group of order $p^m$, $p$ being a prime number, contains a complete set of $p^{m-2}$ conjugate subgroups of order $p$ and also an abelian subgroup of index $p$ then this subgroup cannot involve more than $p$ invariants and when it involves $p$ invariats it must be of type* $(1, 1, 1, \cdots)$.

When such a subgroup is cyclic and $p > 2$ it results directly from the fact that its group of isomorphisms is cyclic that $m$ cannot exceed 3. There is therefore a marked difference between the cases when $p = 2$ and when $p > 2$ as regards such a cyclic subgroup. In particular, when $p = 3$ and this subgroup is cyclic, $G$ is either of order 9 or the non-abelian group of order 27 which involves operators of order 9. When this subgroup is of type $(1, 1, 1)$ $G$ is the group of order 81 whose only abelian subgroup of order 27 is of this type. On the other hand, when this subgroup has two invariants there is no upper limit to the possible orders of $G$ but the ratio of these two invariants can not exceed 3 in view of the fact that if this ratio would exceed 3, $G$ would involve a characteristic subgroup having for one of its invariants 3 and for the other a number larger than 9. Such a subgroup can obviously not involve an automorphism of order 3 under which only three of its operators are invariant.

# TOPOLOGICAL INVARIANCE OF SUB-COMPLEXES OF SINGULARITIES.*

By ARTHUR B. BROWN.

1. *Introduction.* We consider the question of topologically invariant sub-sets of a given complex.† In the first part of the paper we consider sub-complexes which are determined from the incidence relations by simply counting, and here obtain results of which the following is a simple example: If a complex $K$ is an $n$-cycle (mod. 2), then so is every complex homeomorphic to $K$. In the second part of the paper, we consider sub-sets defined in simple terms, but such that it is not known at present how to determine them in all cases from the incidence relations. Given two homeomorphic complexes $K$ and $K'$, in terms of these sub-sets we state a necessary and sufficient condition that the image of a given point of $K$ be on a cell of $K'$ of dimension not exceeding $i$, under all possible homeomorphisms between $K$ and $K'$.

The proofs depend on the Brouwer theorem of invariance of regionality.

*Cells are taken homeomorphic to simplexes,* as defined in Veblen's Analysis Situs.‡

2. *Sub-complexes determined by counting.* Given a complex $K$, let $K_n$ be another name for $K$, and $K_i$ the sub-complex of $K_{i+1}$ consisting of all the $i$-cells, and their boundaries, of $K_{i+1}$ that are incident each with a number of $(i+1)$-cells of $K_{i+1}$ not equal to 2. We remark that we might also have $K_i$ include every $i$-cell, and boundary, of $K$ which is not incident with any cells of $K$ of higher dimensions; and all the succeeding work goes through.

Preparatory to proving the topological invariance of the complexes $K_i$, we prove two lemmas.

LEMMA 1. *Given a complex $K$ and an $(n-1)$-cell $a_{n-1}$ incident with $m$ of its $n$-cells; then if $P$ is a point on $a_{n-1}$ having a neighborhood on $K$ homeomorphic to an $n$-cell, $m$ must equal 2.*

*Proof.* In each of the three other possible cases, $m > 2$, $m = 0$, $m = 1$,

* Presented to the Society, June 13, 1931.

† Analysis situs terminology is as defined in S. Lefschetz's "Topology," *Colloquium Series*, Vol. 12, New York, 1930.

‡ O. Veblen, "Analysis Situs," *The Cambridge Colloquium*, second edition, New York, 1931.

we can easily obtain a contradiction, by use of the theorems of invariance of regionality * and dimensionality.† Therefore $m$ must equal 2.‡

LEMMA 2. *Let $K$ and $K'$ be homeomorphic $n$-complexes, and $b_{n-1}$ an $(n-1)$-cell of $K'$ which is incident with $m$ $n$-cells of $K'$, $m \neq 2$. Then under any homeomorphism $H$ between $K$ and $K'$, $b_{n-1}$ is mapped on cells of $K$ of dimensions not exceeding $n-1$; and, neighboring any point $P'$ on it, has images on at least one $(n-1)$-cell of $K$. Further, if $Q'$ on $b_{n-1}$ corresponds to $Q$ on an $(n-1)$-cell $a_{n-1}$ of $K$, then $a_{n-1}$ and its incident $n$-cells correspond to $b_{n-1}$ and its incident $n$-cells in $K'$, in one-to-one manner, neighboring $Q$ and $Q'$ respectively.*

*Proof.* Let $r$ be the maximum of the dimensions of the cells of $K$ on which $b_{n-1}$, neighboring $P'$, is mapped. Then if $Q'$ on $b_{n-1}$ maps on $Q$ of $a_r$, there must be an $(n-1)$-cell, image of a neighborhood of $Q'$ on $b_{n-1}$, on a neighborhood of $Q$ on $a_r$. By the invariance of regionality, $r$ must therefore be at least $n-1$. Since $m \neq 2$, it follows from Lemma 1 that $r \neq n$, hence $r$ must equal $n-1$, as was to be proved.

To prove the final conclusion of the lemma, let $Q'$ on $b_{n-1}$ have as image $Q$ on $a_{n-1}$ of $K$, and consider the map on $K$ of a neighborhood of $Q'$ on $K'$, say composed of cells one each on $b_{n-1}$ and on the incident $n$-cells. According to the first part of the lemma, the neighborhood of $Q'$ on $b_{n-1}$ must map on a set neighboring $Q$ on $a_{n-1}$, hence, by the invariance of regionality, on a neighborhood of $Q$ on $a_{n-1}$.

Let $N'_n$ be one of the $n$-cells forming the neighborhood of $Q'$ on $K'$. The image $N_n$ of $N'_n$ must lie on just one of the $n$-cells of $K$ incident with $a_{n-1}$. For, if it had points $A$ and $B$ on two different $n$-cells, we could join the respective image-points $A'$ and $B'$ by a curve on $N'_n$ and, taking the image curve on $K$, let $C$ be the first point on that curve, when passing from $A$ to $B$, which did not lie on the $n$-cell on which $A$ lay. Then $C$ could not lie on an $n$-cell, hence would be on $a_{n-1}$. As the corresponding point $C'$ would lie on $N'_n$, it would follow from Lemma 1 that $a_{n-1}$ must be incident with just two $n$-cells of $K$. Then, considering the homeomorphism of the neighborhoods of $Q'$ and $Q$, it would follow, again from Lemma 1, that $b_{n-1}$ must be incident with just two

* L. E. J. Brouwer, "Beweis der Invarianz des $n$-dimensionalen Gebiets," *Mathematischen Annalen*, Vol. 71 (1912), pp. 304-313. Cf. Lefschetz, *loc. cit.*, page 100.

† L. E. J. Brouwer, "Beweis der Invarianz der Dimensionszahl," *Mathematischen Annalen*, Vol. 70 (1911), pp. 161-165. Cf. Lefschetz, *loc. cit.*, page 99. The invariance of dimensionality is of course an immediate corollary of the invariance of regionality.

‡ Cf. proof of invariance of simple circuit, Lefschetz, *loc. cit.*, page 101.

$n$-cells of $K'$, contrary to hypothesis. Hence $N_n$ can lie on only one of the $n$-cells of $K$ incident with $a_{n-1}$.

Further, two different $N'_n$ cannot map on the same $n$-cell of $K$, for in that case we could apply the argument just given, with the rôles of $K$ and $K'$ interchanged, and again obtain a contradiction.

Since under $H$ a neighborhood of $Q$ on $K$ must correspond to a neighborhood of $Q'$ on $K'$, we therefore conclude that the final conclusion of Lemma 2 is valid, and the proof is complete.

The following theorem is a special case of Theorem 6 below.

THEOREM 3. *Let $K$ and $K'$ be homeomorphic $n$-complexes, and $M$ and $M'$ their respective sub-complexes composed of the $n$-cells and their boundaries. Then under any homeomorphism between $K$ and $K'$, $M$ corresponds to $M'$, and the boundary (mod. 2) of $M$ corresponds to the boundary (mod. 2) of $M'$.*

*Proof.* The first conclusion is proved easily by use of the theorem of invariance or regionality. The second conclusion is also proved easily, by use of Lemma 2 and the fact that an $(n-1)$-cell on the boundary of an $n$-complex is incident with an odd number of $n$-cells, hence not just 2. We give no further details.

COROLLARY 4. *If $K$ and $K'$ are homeomorphic complexes, then $K$ is an $n$-cycle (mod. 2) if and only if $K'$ is an $n$-cycle (mod. 2).**

THEOREM 5. *Let $K$ and $K'$ be two homeomorphic complexes, and $K_i$, $K'_i$, $i = 0, 1, \cdots, n$, the respective sub-complexes defined at the beginning of the section. Then, under any homeomorphism $H$, $K_i$ and $K_i'$ correspond as point sets.*

*Proof.* Assume the theorem proved for $K_{n-1}, \cdots, K_{i+1}$. The theorem for $K_i$ is then an easy consequence of Lemma 2, taking the $n$ of Lemma 2 to be $i+1$ for the present application. As the case $i = n-1$ requires no assumption, it follows that the theorem is true.

THEOREM 6. *Given a complex $K$, let $A$ be the sub-complex determined by the $i$-cells satisfying the following conditions. Each $i$-cell of $A$ is an $i$-cell of $K_i$ (defined above), and is incident with a specified number of $(i+1)$-cells of $K_{i+1}$; each of the latter $(i+1)$-cells is incident with a specified number of $(i+2)$-cells of $K_{i+2}$ (not necessarily the same number in every case); $\cdots$;*

* The corresponding theorem for the $n$-circuit was proved by Veblen, *loc. cit.*, Chap. III (first or second edition); for the simple $n$-circuit by Lefschetz, *loc. cit.*, page 101.

*and so on, till finally we have $(n-1)$-cells of $K_{n-1}$ incident with specified numbers of n-cells of K. Let K′ be a complex homeomorphic to K, and A′ the sub-complex of K′ defined as described above for A and K, and with the same sets of numbers specified. Then, under any homeomorphism H between K and K′, A and A′ must correspond.*

To prove this theorem, we assume it proved for the sub-complexes similarly defined, but starting with cells of dimension $(i+1)$, and then proceed to prove it for the case stated. The result follows easily upon use of Lemma 2. Hence, as in the preceding proof, the theorem is valid.

3. *Loci of irregular points.* A point $P$ on a complex $K$ will be said to be *p-regular,* or have regularity of order $p$, if $p$ is the largest integer such that a neighborhood of $P$ on $K$ can be covered by a set of cells all of dimensions at least $p$; where the set is non-singular on $K$, and the cells are incident with each other as in a complex. The cells need not be cells of $K$. The determination of the order of regularity of a given point of $K$ in terms of the incidence relations between the cells of $K$ would involve the solution of some of the outstanding problems of analysis situs; but it is done easily in many cases.

We define $J_i$, $i = 0, 1, \cdots, n$, to be the locus of all points on $K$ of regularity $i$ or less.

THEOREM 7. *The locus $J_n$ is all of K, and $J_i$ is a sub-complex of $J_{i+1}$, of dimension at most i, $i = 0, 1, \cdots, n-1$.*

The theorem follows easily from the definition, and we omit the proof.

THEOREM 8. *Given K and K′ homeomorphic complexes, and $J_i$ and $J'_i$ their respective sub-complexes of points of regularity i or less. Then $J_i$ and $J'_i$ correspond under any homeomorphism between K and K′. Furthermore, a necessary and sufficient condition that a given point P′ of K′ be mapped on a cell of K of dimension not exceeding i, under all possible homeomorphisms H between K and K′, is that P′ be on $J'_i$.*

*Proof.* The first conclusion is a consequence of the fact that the definition of $J_i$ is purely topological. The sufficiency of the condition in the last conclusion follows from the first part of the theorem. We now prove the necessity.

Let $P$ be the image of $P'$ on $K$. Since $P'$ is not on $J'_i$, $P$ is not on $J_i$, and hence has a neighborhood on $K$ composed of cells of dimensions at least $i+1$ (not necessarily cells of $K$, of course). Let $d_r$ denote the cell of that neighborhood on which $P$ lies. Since the dimension $r$ of $d_r$ is at least $i+1$,

$d_r$ must contain a point $Q$ as near to $P$ as we like, which does not lie on any cell of $K$ of dimension $i$ or less. This is an easy consequence of the invariance of regionality. It is an easy matter now to define a new homeomorphism $H_1$ between $K$ and $K'$, which maps $P'$ on $Q$, but is identical with $H$ except in the neighborhoods of $P$ (including $Q$) and $P'$. We need consider $H_1$ only in these neighborhoods: First we define $H_1$ over $d_r$, so as to be identical with $H$ near the boundary of $d_r$. Next we can define it also over the $(r+1)$-cells of the neighborhood, so as to be continuous over $d_r$ and these $(r+1)$-cells, and identical with $H$ near the (point-set) boundary. Continuing in this way, $H_1$ is finally determined for the entire neighborhood. Since $Q$ is on an $r$-cell of $K$, with $r \geqq i+1$, it follows that the final conclusion of the theorem is valid.

THEOREM 9. *The complex $K_i$ is a sub-complex of $J_i$.*

*Proof.* Since $K_i$ and $J_i$ are both sub-complexes of $K$, and $K_i$ is determined by its $i$-cells, it is sufficient to prove that if $P$ is a point on an $i$-cell of $K_i$, then $P$ lies on $J_i$. Now if $P$ were not on $J_i$, we could cover a neighborhood $N$ of $P$ on $K$ by cells of dimensions at least $i+1$, by definition of $J_i$. Now Theorem 5 holds in the small, as follows from the proofs; hence, relative to $N$, $K_i$ must coincide with an $i$-dimensional sub-set of the cells forming $N$. As the cells forming $N$ would be of dimensions at least $i+1$, $K_i$ could therefore have no points on $N$, a contradiction to the fact that $P$ on $N$ is on $K_i$. Hence $P$ must be on $J_i$, and the theorem is proved.

THEOREM 10. *Theorem 6 holds with $K_i$ replaced by $J_i$.*

*Proof.* The same proof applies as in the case of Theorem 6, except for the case of an $i$-cell of $J_i$ which is incident with zero or two $(i+1)$-cells of $J_{i+1}$. Here we use the fact that the given $i$-cell must be mapped on $J'_i$ (Theorem 8), and then Lemma 2 will suffice for completing the proof of the induction. Thus Theorem 10 is valid.

4. *Concluding remarks.* The sub-complexes $K_i$ might easily be made more inclusive. For example, we might let $K_i$ contain all $i$-cells, and their boundaries, each of which is incident with at least one $(i+1)$-cell of each of the following two classes: (a) $(i+1)$-cells which are not incident with any $(i+2)$-cells; (b) $(i+1)$-cells each of which is incident with at least one $(i+2)$-cells. All the theorems of the paper would remain valid.

Even under an extended definition of $K_i$, $K_i$ would not in all cases include

all of $J_i$. Consider, for example, the complex obtained as the join of a point $P$ with a torus.* The point $P$ is not on $K_0$, $K_1$ or $K_2$, but is easily proved to be on $J_0$, $J_1$ and $J_2$.

Finally, we observe that if we consider a complex composed of "cells" each of which, with its boundary, is homeomorphic to an ordinary manifold with regular boundary,† hence not necessarily homeomorphic to a simplex, all the work of the paper goes through unchanged, and all the theorems are valid.

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* Cf. Veblen, *loc. cit.*, page 96.

† The combinatorial manifold with regular boundary is defined by Lefschetz, *loc. cit.*, page 120. When we limit ourselves to ordinary manifolds, the same definitions apply, except that all cells, spheres, etc., appearing in the definitions of Lefschetz are required to be homeomorphic to ordinary spheres, simplexes, etc. Cf. Veblen, *loc. cit.*, pp. 88 and 92.

# A DETERMINATION OF THE TYPES OF PLANAR CREMONA TRANSFORMATIONS WITH NOT MORE THAN 9 *F*-POINTS.

By MILDRED E. TAYLOR.

1. *Introduction.* The types of Cremona transformations can be classified according to the order $n$ or according to the number $\rho$ of the $F$-points. Coble, [3, p. 83], remarks "that perhaps the latter classification is the more fundamental." Roughly speaking, the range of usefulness of a transformation $T$ is inversely proportional to the number $\rho$ of the $F$-points. For, unless the $F$-points of $T$ can be placed at significant points of a given figure, the image is more complicated rather than more simple.

Cremona[6,7] gave a list of transformations for $n = 2$ to $n = 10$. Guccia[8] gave algebraic expressions for the types and their inverses as listed by Cremona. Mlodziejowski [10] listed types for $n = 2$ to $n = 21$. Bianchi [1] by multiplying two De Jonquières transformations obtained the expression for the order $n$ involving 3 parameters. Palatini[12] by multiplying three De Jonquières transformations obtained a new transformation of order $n$ whose expression involves 7 parameters.* He also multiplied four De Jonquières transformations to obtain a new one whose order $n$ involves 15 parameters. In general, if $k$ De Jonquières transformations be compounded the expression for the order $n$ of the new transformation involves $(2^k - 1)$ parameters. Miss Hudson[9] gave types for $n = \alpha\mu - \beta$, where $\beta = 2\gamma + \epsilon$, $\epsilon = 0, 1$. If the number of $F$-points is 9, there exists an infinite number of transformations. However, in any of the above types when the number of $F$-points is limited to 9, only a finite number of transformations is obtained. Montesano[11] derived the semi-symmetric types for $\rho = 9$ and obtained independent expressions for each type involving 1 parameter. This paper shows that these types are all related.

In this paper I have obtained explicit algebraic expressions for the integers $n$, $r_i$, $s_j$, $\alpha_{ji}$ attached to every planar Cremona transformation with not more than 9-points, say at $P_9^2$, i. e. $p_1, \cdots, p_9$. The method is that used throughout the literature—the composition of known types. These algebraic expressions are classified into *seven* distinct types.

* Miss Hudson [9] states that the expression which Palatini obtained for $n$ by compounding three De Jonquières transformations involves 6 parameters and by compounding four De Jonquières transformations involves 16 parameters.

2. *The Types in the Invariant Subgroup $i_{9,2}$ of the Linear Group $g_{9,2}$.* The product of two Bertini involutions $E_2$ and $E_1$ with simple points at $p_2$, $p_1$ respectively is called $C_{2,1}$. The semi-symmetric transformation of order 37 with eight 13-fold points and one 4-fold point is called $D_9$ when the 4-fold point is at $p_9$. The general element [5, p. 375] of the abelian subgroup of the invariant subgroup $i_{9,2}$ of the linear group $g_{9,2}$ is given by

$$D_2^{\nu_2}D_3^{\nu_3}\cdots D_9^{\nu_9}C_{2,1}^{\rho_2}C_{3,1}^{\rho_3}\cdots C_{9,1}^{\rho_9}, \tag{1}$$

where $\nu_i$ is any integer, positive or negative, and $\rho_i = 0, 1, 2$ with $\Sigma\rho_i \equiv 0 \bmod 3$ $(i = 2, 3, \cdots, 9)$.

By direct multiplication of the factors of (1), explicit algebraic expressions are obtained for $n$, $r_i$, $s_j$, $\alpha_{ji}$ of Type I.

The following notation is used throughout for the elementary symmetric functions of the $\nu$'s and the $\rho$'s:

$$\nu = \Sigma_2^9\nu_i,\ \nu' = \Sigma_2^9\nu_i\nu_j;\quad \rho = \Sigma_2^9\rho_i,\ \rho' = \Sigma_2^9\rho_i\rho_j;\quad \sigma = \Sigma_2^9\nu_i\rho_i. \tag{2}$$

Also let

$$\gamma = 4\nu^2 + 4\rho^2 - 9\nu' - 4\rho' - 6\sigma. \tag{3}$$

TYPE I.

$$D_2^{\nu_2}D_3^{\nu_3}\cdots D_9^{\nu_9}C_{2,1}^{\rho_2}C_{3,1}^{\rho_3}\cdots C_{9,1}^{\rho_9}.$$

With $i, j = 1, 2, \cdots, 9$ $(i \neq j)$ and $k = 2, 3, \cdots, 9$, let

$$\delta_1 = 2\rho,\quad \delta_k = 3\nu_k - 2\rho_k.$$

Then

$$\begin{aligned} &n = 9\gamma + 1,\quad r_i = 3\gamma + \nu - 3\delta_i,\quad s_j = 3\gamma - \nu + 3\delta_j,\\ &\alpha_{ii} = \gamma - 1,\quad \alpha_{ji} = \gamma + \delta_j - \delta_i. \end{aligned} \tag{4}$$

The remaining elements of $i_{9,2}$ [5, p. 375] are obtained by taking the product of $E$, and (1).

TYPE II.

$$E_1D_2^{\nu_2}D_3^{\nu_3}\cdots D_9^{\nu_9}C_{2,1}^{\rho_2}C_{3,1}^{\rho_3}\cdots C_{9,1}^{\rho_9}.$$

With $i$, $j$, $k$ and $\delta_1$, $\delta_k$ as in Type I let

$$\gamma' = \gamma - \nu + 4\rho + 2,\quad \delta'_1 = \delta_1 + 2,\quad \delta'_k = \delta_k.$$

Then

$$(5)\quad \begin{aligned} &n = 9\gamma' + 3\nu - 1, \quad r_i = 3\gamma' + 2\nu - 3\delta_i, \quad s_j = 3\gamma' + 2\nu - 3\delta'_j \\ &\alpha_{ii} = \gamma' + \nu - 2\delta'_i + 1, \quad \alpha_{ji} = \gamma' + \nu - \delta'_i - \delta'_j. \end{aligned}$$

3. *The Remaining Types in the Linear Group $g_{9,2}$.* The factor group $f_{9,2}$ of the group $g_{9,2}$ [5, p. 373] with respect to $i_{9,2}$ is simply isomorphic with the Cremona group $G_{8,2}$ [4, p. 15; 5, p. 349-350]. A study of the transformations in $G_{8,2}$, which is a finite group, shows that each one can be obtained from an element of $i_{9,2}$ by not more than two quadratic transformations. Each type of $i_{9,2}$ is multiplied by a single quadratic, two unrelated quadratic (no $F$-points in common), or two related quadratic transformations (one $F$-point in common). There are exactly three different elements of the abelian subgroup of $i_{9,2}$ that will give the same Cremona transformation when multiplied by two unrelated quadratic transformations. Exactly three elements of the non-abelian part of $i_{9,2}$ when multiplied by two unrelated quadratic transformations give the same Cremona transformation. A Cremona transformation which is the product of an element of the abelian subgroup of $i_{9,2}$ and two related quadratic transformations is also the product of some element of the non-abelian part of $i_{9,2}$ and two related quadratic transformations. Hence, there are only *seven* distinct types of planar Cremona transformations.

Since the $F$-points can be permuted without altering the transformation, the quadratic transformations $A_{123}$, $A_{456}$, $A_{145}$ with $F$-points at $p_1$, $p_2$, $p_3$; $p_4$, $p_5$, $p_6$; $p_1$, $p_4$, $p_5$, respectively are used to simplify the notation.

## Type III.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123}.$$

With $i, j = 1, 2, 3$ $(i \neq j)$ and $k, l = 4, 5, \cdots, 9$ $(k \neq l)$, let

$$\epsilon_{123} = \nu - (\delta_1 + \delta_2 + \delta_3).$$

Then

$$(6)\quad \begin{aligned} &n = 9\gamma - 3\epsilon_{123} + 2, \\ &r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1, \quad r_k = 3\gamma + \nu - 3\delta_k, \\ &s_j = 3\gamma - \nu - \epsilon_{123} + 3\delta_j + 1, \quad s_l = 3\gamma - \nu - \epsilon_{123} + 3\delta_l, \\ &\alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - 1, \quad \alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \\ &\alpha_{jk} = \gamma - \epsilon_{123} + \delta_j - \delta_k, \quad \alpha_{li} = \gamma + \delta_l - \delta_i, \quad \alpha_{lk} = \gamma + \delta_l - \delta_k. \end{aligned}$$

## Type IV.

$$E_1 D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123}.$$

With $\gamma'$ and $\delta'$'s as in Type II, and $i, j, k, l$ as in Type III, let

$$\epsilon'_{123} = \nu - (\delta'_1 + \delta'_2 + \delta'_3).$$

Then

$$\begin{aligned}
&n = 9\gamma' + 3\nu - 3\epsilon'_{123} - 2,\\
&r_i = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\delta'_i - 1, \quad r_k = 3\gamma' + 2\nu - 3\delta'_k,\\
&s_j = 3\gamma' + 2\nu - \epsilon'_{123} - 3\delta'_j - 1, \quad s_l = 3\gamma' + 2\nu - \epsilon'_{123} - 3\delta'_l,\\
&\alpha_{ii} = \gamma' + \nu - \epsilon'_{123} - 2\delta'_i, \quad \alpha_{kk} = \gamma' + \nu - 2\delta'_k + 1,\\
&\alpha_{ji} = \gamma' + \nu - \epsilon'_{123} - \delta'_i - \delta'_j - 1, \quad \alpha_{li} = \gamma' + \nu - \epsilon'_{123} - \delta'_l - \delta'_i,\\
&\alpha_{jk} = \gamma' + \nu - \delta'_j - \delta'_k, \quad \alpha_{lk} = \gamma' + \nu - \delta'_l - \delta'_k.
\end{aligned} \tag{7}$$

## Type V.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123} A_{456}.$$

With $i, j = 1, 2, 3$ $(i \neq j)$, $k, l = 4, 5, 6$ $(k \neq l)$, and $m, t = 7, 8, 9$ $(m \neq t)$, let $\epsilon_{456} = \nu - (\delta_4 + \delta_5 + \delta_6)$. Then

$$\begin{aligned}
&n = 9\gamma - 6\epsilon_{123} - 3\epsilon_{456} + 4,\\
&r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1, \quad r_k = 3\gamma + \nu - 3\epsilon_{123} - 3\epsilon_{456} - 3\delta_k + 2,\\
&r_m = 3\gamma + \nu - 3\delta_m, \quad s_j = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_j + 2,\\
&s_l = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_l + 1, \quad s_t = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_t,\\
&\alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - \epsilon_{123} - \epsilon_{456}, \quad \alpha_{mm} = \gamma - 1,\\
&\alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \quad \alpha_{li} = \gamma - \epsilon_{123} + \delta_k - \delta_i,\\
&\alpha_{ti} = \gamma - \epsilon_{123} + \delta_t - \delta_i, \quad \alpha_{jk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_j - \delta_k + 1,\\
&\alpha_{lk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_l - \delta_k + 1, \quad \alpha_{tk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_t - \delta_k,\\
&\alpha_{jm} = \gamma + \delta_j - \delta_m, \quad \alpha_{lm} = \gamma + \delta_l - \delta_m, \quad \alpha_{tm} = \gamma + \delta_t - \delta_m.
\end{aligned} \tag{8}$$

## Type VI.

$$E_1 D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123} A_{456}.$$

With $i, j, k, l, m, t$ as in Type V and $\epsilon'_{123}$ as in Type IV, let

$$\epsilon'_{456} = \nu - (\delta'_4 + \delta'_5 + \delta'_6).$$

Then

$$\begin{aligned}
&n = 9\gamma' + 3\nu - 6\epsilon'_{123} - 3\epsilon'_{456} - 4,\\
&r_i = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\delta'_i - 1, \quad r_k = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\epsilon'_{456} - 3\delta'_k - 2,\\
&r_m = 3\gamma' + 2\nu - 3\delta'_m, \quad s_j = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_j - 2,\\
&s_l = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_l - 1, \quad s_t = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_t,\\
&\alpha_{ii} = \gamma' + \nu - \epsilon'_{123} - 2\delta'_i, \quad \alpha_{kk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - 2\delta'_k,\\
&\alpha_{mm} = \gamma' + \nu - 2\delta'_m + 1, \quad \alpha_{ji} = \gamma' + \nu - \epsilon'_{123} - \delta'_i - \delta'_j - 1,\\
&\alpha_{li} = \gamma' + \nu - \epsilon'_{123} - \delta'_l - \delta'_i - 1, \quad \alpha_{ti} = \gamma' + \nu - \epsilon'_{123} - \delta'_t - \delta'_i - 1,
\end{aligned} \tag{9}$$

$$\alpha_{jk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_j - \delta'_k - 1,$$
$$\alpha_{lk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_l - \delta'_k - 1,$$
$$\alpha_{tk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_t - \delta'_k, \quad \alpha_{jm} = \gamma' + \nu - \delta'_j - \delta'_m,$$
$$\alpha_{lm} = \gamma' + \nu - \delta'_l - \delta'_m, \quad \alpha_{tm} = \gamma' + \nu - \delta'_t - \delta'_m.$$

## TYPE VII.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123} A_{145}.$$

With $i, j = 2, 3$ $(i \neq j)$, $k, l = 4, 5$ $(k \neq l)$ and $m, t = 6, 7, 8, 9$ $(m \neq t)$ and $\epsilon_{123}$ as in Type III, let $\epsilon_{145} = \nu - (\delta_1 + \delta_4 + \delta_5)$. Then

$$n = 9\gamma - 3\epsilon_{123} - 3\epsilon_{145} + 3,$$
$$r_1 = 3\gamma + \nu - 3\epsilon_{123} - 3\epsilon_{145} - 3\delta_1 + 2, \quad r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1,$$
$$r_k = 3\gamma + \nu - 3\epsilon_{145} - 3\delta_k + 1, \quad r_m = 3\gamma + \nu - 3\delta_m,$$
$$s_1 = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_1 + 2, \quad s_j = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_j + 1,$$
$$s_l = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_l + 1, \quad s_t = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_t,$$
$$\alpha_{11} = \gamma - \epsilon_{123} - \epsilon_{145} + 1, \quad \alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - \epsilon_{145}, \quad \alpha_{mm} = \gamma - 1,$$
(10) $$\alpha_{j1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_j - \delta_1 + 1, \quad \alpha_{l1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_l - \delta_1 + 1,$$
$$\alpha_{t1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_t - \delta_1, \quad \alpha_{1i} = \gamma - \epsilon_{123} + \delta_1 - \delta_i + 1,$$
$$\alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \quad \alpha_{li} = \gamma - \epsilon_{123} + \delta_l - \delta_i,$$
$$\alpha_{ti} = \gamma - \epsilon_{123} + \delta_t - \delta_i, \quad \alpha_{1k} = \gamma - \epsilon_{145} + \delta_1 - \delta_k + 1,$$
$$\alpha_{jk} = \gamma - \epsilon_{123} + \delta_j - \delta_k, \quad \alpha_{lk} = \gamma - \epsilon_{145} + \delta_l - \delta_k + 1,$$
$$\alpha_{tk} = \gamma - \epsilon_{145} + \delta_t - \delta_k, \quad \alpha_{1m} = \gamma + \delta_1 - \delta_m, \quad \alpha_{jm} = \gamma + \delta_j - \delta_m,$$
$$\alpha_{lm} = \gamma + \delta_l - \delta_m, \quad \alpha_{tm} = \gamma + \delta_t - \delta_m.$$

4. *Bertini L-Curves with not more than 9 Multiple Points.* Since the $P$-curves of the planar Cremona transformations are Bertini $L$-curves [[2]; [4], p. 22], the algebraic expressions for $r_i$, $s_j$, $\alpha_{ji}$ in (4), (5) $\cdots$ (10) give expressions for the order $t$ and multiplicity $t_i$ of all the Bertini $L$-curves with not more than 9 multiple points. Each such Bertini curve occurs at least once, but may occur more often in this set of expressions.

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# ON THE REPRESENTATIVE SPACE $S_9$ OF THE PLANE CUBICS.*

By J. YERUSHALMY.†

In the study of the plane curves of order $n$ it is often convenient to introduce the terminology of projective geometry of hyperspace by representing the curves by points (or hyperplanes) of a space $S_N$ of $N = n(n+3)/2$ dimensions. All the curves satisfying an algebraic condition are then represented by the points of a certain algebraic spread in $S_N$. Thus all the curves having a double-point are given by a spread of $N-1$ dimensions, those having two double-points by a spread of $N-2$ dimensions, etc.

The representative space $S_5$ of the conics has been studied extensively by Veronese,‡ Segre,§ Study,¶ and others. In what follows some properties of the $S_9$ of the plane cubics are found. We obtain the orders of the more important spreads and their multiplicities with respect to each other, and the linear spaces on them with special reference to the surface $F_2^9$ representing cubics which degenerate into triple lines. A more detailed study is made of the $F_2^9$ by determining the curves on it, its tangent planes and hyperplanes, as well as its double and triple tangent hyperplanes. Most of the results are then extended to the general representative space $S_N$ $[N = n(n+3)/2]$ of the plane $n$-ics.

We denote by $V_r^s$ or $F_r^s$ a spread of $r$ dimensions of order $s$; while a linear space of dimension $k$ is denoted by $S_k$ or $R_k$.

A $V_8^n$ and $S_8$ in $S_9$ are called *hypersurface* and *hyperplane* respectively; a $V_2^n$, a surface.

For brevity we refer to a point of $S_9$ as the cubic of the plane which it represents.

1. *The orders of the spreads.* The cubics of a plane $\pi$ depend, as is known, on one absolute parameter, which is the constant cross-ratio of the 4 tangents which may be drawn to the cubic from a point on it, and which

* Presented at the Chicago Meeting of the American Mathematical Society April 4, 1931.

† National Research Fellow, University of Illinois.

‡ Veronese, "La superficie omaloide, etc.," *Memorie dell' Accademia dei Lincei* (19)3, 1883-4.

§ Segre, "Considerazioni intorno alla geometria della coniche di un piane," *Atti della R. Accademia delle Scienze di Torino*, 20, 1885.

¶ Study, "Über Geometrie der Kegelschnitte," *Mathematische Annalen*, Vol. 27.

is called the *modulus* of the cubic. In fact, if $\alpha$ is the value of this cross-ratio for a cubic, then

$$(1.1) \qquad J = \frac{4(\alpha^2 - \alpha + 1)^3}{(1+\alpha)^2(2-\alpha)^2(1-2\alpha)^2} = \frac{S^3}{T^2}$$

is the only absolute rational invariant for the cubic, where $S$ and $T$ are the two relative invariants of the cubic of degrees 4 and 6 respectively in the coefficients.

That a cubic be of given modulus is therefore equivalent to one condition and we have in $S_9$ $\infty^1$ hypersurfaces corresponding to the $\infty^1$ values of the parameter $J$. The points of one of these hypersurfaces correspond to all the cubics in $\pi$ having the same modulus. Corresponding to the values $\infty$, 0, 1 of $J$ $[\alpha = -1; -\epsilon(\epsilon^3 = 1); 1]$ there are three hypersurfaces whose points correspond to harmonic, equianharmonic, and nodal cubics respectively.

We list below the spreads in $S_9$ in which we will be interested.

a. $\infty^1$ $F_8$'s of cubics of equal generic modulus.
b. a $V_8$ of nodal cubics.
c. an $F_8$ of harmonic cubics.
d. an $F_8$ of equianharmonic cubics.
e. a $V_7$ of cubics having 2 double-points (degenerating into a line and a conic).
f. an $F_7$ of cuspidal cubics.
g. a $V_6$ of cubics having 3 double-points (composed of 3 lines).
h. an $F_6$ of cubics having a tacnode (degenerating into a conic and a tangent line).
i. an $F_5$ of cubics having a triple-point (degenerating into 3 lines of a pencil).
j. an $F_4$ of cubics composed of a double line and a simple line.
k. an $F_2$ of cubics degenerating into a triple line.

We recall that the order of a $V_r^s$ in $S_N$ is the number, $s$, of points in which it is met by an arbitrary $S_{N-r}$. Thus to find the order of any of the above spreads $V_k$, we look for the number of cubics having the corresponding singularity which are contained in an arbitrary linear system $\infty^{9-k}$ of cubics.

a. Since $J$ (1.1) involves the coefficients of the cubic to the 12-th degree, there are in an arbitrary pencil of cubics 12 cubics of a given generic modulus, and *each of the hypersurfaces is an* $F_8^{12}$.

b. In the same way *the hypersurface of nodal cubics is a* $V_8^{12}$.

c. The condition that a cubic be harmonic is $T = 0$, which involves the coefficients to the 6-th degree. *This hypersurface is therefore an* $F_8^6$.

d. For an equianharmonic cubic $S = 0$, involving the coefficients to the 4-th degree. *This hypersurface is an* $F_8^4$.

In an arbitrary net of curves of order $n$ there are $\frac{3}{2}(n-1)(n-2) \times (3n^2 - 3n - 11)$ curves having two double-points and $12(n-1)(n-2)$ cuspidal curves.* For $n = 3$ we obtain

e. *The spread of cubics degenerating into a conic and a line is a* $V_7^{21}$.

f. *The spread of cuspidal cubics is an* $F_7^{24}$.

Caporali † proves that in a linear system $\infty^3$ of curves of order $n$, of genus $p$ with $\sigma$ base-points and of grade $N$ there are

$$\tfrac{1}{6}(N + 4p + \sigma)^3 - (N + 4p + \sigma)(2N + 41p + \sigma) - \tfrac{1}{6}(175N - 3434P + 223\sigma) + 106$$

curves having 3 double-points and $2(64p - 7N - 6\sigma + 20)$ curves having a tacnode. For $n = 3$, $p = 1$, $\sigma = 0$, $N = 9$ we have

g. *The spread of cubics composed of 3 lines of a triangle is a* $V_6^{15}$.

h. *The spread of cubics degenerating into a conic and a tangent line is an* $F_6^{42}$.

The order of $F_5$ of $i$ is the number of cubics with a triple point contained in a generic linear system $\infty^4$ of cubics

$$(1.2) \quad \lambda_1\phi_1(x_1x_2x_3) + \lambda_2\phi_2(x_1x_2x_3) + \lambda_3\phi_3(x_1x_2x_3) + \cdots + \lambda_5\phi_5(x_1x_2x_3) = 0.$$

The condition for a triple-point is that all the derivatives of second order vanish:

$$(1.3) \qquad \sum_{a=1}^{5} \lambda_a \frac{\partial^2\phi_a}{\partial x_1^2} = 0, \qquad \sum_{a=1}^{5} \lambda_a \frac{\partial^2\phi_a}{\partial x_2^2} = 0, \cdots \sum_{a=1}^{5} \lambda_a \frac{\partial^2\phi_a}{\partial x_2 \partial x_3} = 0.$$

The number of such curves in the system (1.2) is given by the order of the matrix of 6 rows and 5 columns

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* Enriques and Chisini, *Lezioni sulla teorie geometria delle equazioni, etc.*, Vol. 22, p. 178.

† Caporali, "Sopra i sistemi lineari triplamante infiniti di curve algebriche piane," *Memorie di Geometria*, Napoli (1888), p. 191-4.

$$(1.4)\qquad \begin{pmatrix} \frac{\partial^2\phi_1}{\partial x_1^2} & \frac{\partial^2\phi_2}{\partial x_1^2} & \cdots & \frac{\partial^2\phi_5}{\partial x_1^2} \\ \frac{\partial^2\phi_2}{\partial x_2^2} & \frac{\partial^2\phi_2}{\partial x_2^2} & \cdots & \frac{\partial^2\phi_5}{\partial x_2^2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2\phi_1}{\partial x_2\partial x_3} & \frac{\partial^2\phi_2}{\partial x_2\partial x_3} & \cdots & \frac{\partial^2\phi_5}{\partial x_2\partial x_3} \end{pmatrix}$$

whose elements are linear in $x_1$, $x_2$, $x_3$. The order of a matrix of $n$ rows and $n+k$ columns in which the order of the element in the $i$-th row and the $j$-th column is $l_j + \lambda_i$ is *

$$(1.5)\qquad H_{k+1} = \mu_{k+1} + \mu_k\nu_1 + \mu_{k-1}\nu_2 + \cdots + \nu_{k+1}$$

where $\mu_i$ and $\nu_i$ are the elementary symmetric functions of degree $i$ formed from $l_1, l_2, \cdots l_{n+k}$ and $\lambda_1, \lambda_2, \cdots \lambda_{n+k}$ respectively. In our case $l_1 = l_2 = \cdots = l_6 = 1$, $\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 0$; $H_2 = \mu_2 = 15$. Hence

i. *The spread of cubics with a triple point is an* $F_5^{15}$.

j1. The order of $V_4$ of $j$ is the number of cubics composed of a double line and a simple line which are contained in a generic linear system $\infty^5$ of cubics

$$(1.6)\qquad \lambda_1\phi_1(x_1x_2x_3) + \lambda_2\phi_2(x_1x_2x_3) + \cdots + \lambda_6\phi_6(x_1x_2x_3) = 0.$$

This system contains $\infty^1$ cubics with a triple point. The locus of the triple points is the sextic curve given by

$$(1.7)\qquad C_6 \equiv \begin{vmatrix} \frac{\partial^2\phi_1}{\partial x_1^2} & \frac{\partial^2\phi_2}{\partial x_1^2} & \cdots & \frac{\partial^2\phi_6}{\partial x_1^2} \\ \frac{\partial^2\phi_1}{\partial x_2^2} & \frac{\partial^2\phi_2}{\partial x_2^2} & \cdots & \frac{\partial^2\phi_6}{\partial x_2^2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2\phi_1}{\partial x_2\partial x_3} & \frac{\partial^2\phi_2}{\partial x_2\partial x_3} & \cdots & \frac{\partial^2\phi_6}{\partial x_2\partial x_3} \end{vmatrix} = 0.$$

This is a general sextic of full genus as can be seen by applying a formula

* This formula was first inferred by Salmon, *Higher Algebra*, and was proved later by F. F. Decker and A. B. Coble. See Coble, "Restricted Systems of Equations," *American Journal of Mathematics*, Vol. 36, p. 410.

given by Coble (*loc. cit.*) which determines the genus of a curve given by a matrix.

Take now an arbitrary line $l$ in the plane $\pi$ of the system (1. 6). Thru any point $P$ on it there passes a linear system $\infty^4$ of cubics of the system, which contains, by the previous result, 15 cubics having a triple point. Each of these cuts $l$ in 2 other points $Q_1$ and $Q_2$. To each point $P$ correspond therefore 30 points $Q$, and every point $Q$ arises from 30 points $P$. We have on $l$ a [30, 30] correspondence with 60 coincidences. These are accounted for by (a) the intersections of $l$ with $C_6$, and (b) by the intersections of $l$ with the double lines of the cubics of the system (1. 6) composed of a double line and a simple line.

a) When $P$ is one of the 6 points of intersection of $l$ with $C_6$, there passes through $P$ one cubic having a triple-point at $P$. If the point $P$ is considered as belonging to one of the three branches of the cubic, two points $Q$ correspond to it belonging to the other two branches of the curve. But we must consider the point $P$ as belonging successively to the three branches of the curve, and hence such a point absorbs $3 \cdot 2 = 6$ coincidences.

b) When $P$ is an intersection of $l$ with the double-line of a cubic of the system composed of a double-line and a simple line, one point $Q$ coincides with it since, as will be seen later, $F_4$ is simple on $F_5{}^{15}$. The number of such cubics in the system is therefore $60 - 6 \cdot 6 = 24$,* and:

j. *The spread of cubics composed of a double-line and a simple line is an* $F_4{}^{24}$.

Corresponding to a linear system $\infty^7$ of cubics there exists a range of curves of class 3 such that every curve of one system is apolar to every curve of the other system. If a cubic which degenerates into a triple-line is apolar to a curve of class 3 it must be on it. Hence the number of cubics composed of a triple line which are contained in a linear system $\infty^7$ of cubics is equal to the number of lines common to a range of curves of class 3 which is 9, and

k. *The spread of cubics composed of a line counted three times is an* $F_2{}^9$.

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* Professor Zariski pointed out to me that a more conclusive proof can be given by establishing parametrically the equation of the correspondence. The reasoning given here is the same as the one used by Chasles, *Compte Rendue*, Vol. 64 (1867), p. 799, in establishing the general formula

$$2\mu(n-1) - \nu = rr'(r'-1) + s(s'-1)$$

of which ours is a special case with $\mu = 15$, $\nu = 0$, $r' = 3$, $r = 6$, $s' = 2$. At the end of the paper the order of this spread is obtained by a different method and may be used as a check on this correspondence.

2. *The multiplicities of the spreads on each other.* If a cubic has a singularity $\alpha$ which is the limit form of another singularity $\beta$, then the spread of cubics $\alpha$ is on the spread of cubics $\beta$ and generally with some multiplicity $\mu \geqq 1$. All the hypersurfaces are given by $S^3 - kT^2 = 0$ and for $F_7^{24}$, $S = T = 0$, it is clear that the $F_7^{24}$, which is the complete intersection of $F_8^6$ and $F_8^4$, is on all the hypersurfaces, and all the spreads of lower dimensions which are on $F_7^{24}$ are therefore also on all the hypersurfaces. The $V_8^{12}$ of nodal cubics contains evidently all the spreads of lower dimensions. It is easily seen that the position of the spreads on one another can be stated thus:

*All four of the hypersurfaces contain $F_7^{24}$ but $V_8^{12}$ alone contains $V_7^{21}$; $V_7^{21}$ contains $V_6^{15}$ and $F_6^{42}$ but $F_7^{24}$ contains only $F_6^{42}$; and $F_6^{42}$, $V_6^{15}$, $F_5^{15}$, $F_4^{24}$ each contain every spread of lower dimension.*

We proceed to determine the multiplicities of most of the spreads on one another. A spread $V_{r_1}^{s_1}$ is of multiplicity $\mu$ on a spread $V_{r_2}^{s_2}$ in $S_N$ if an arbitrary $S_{N-r_2}$ on a generic point of $V_{r_1}^{s_1}$ cuts $V_{r_2}^{s_2}$ in $s_2 - \mu$ additional points.

The multiplicities of all the spreads of lower dimension on the hypersurfaces are obtained by direct calculation. We take a pencil of cubics $\phi_1 + \lambda\phi_2 = 0$ and let $\phi_1$ take on respectively the form of the various spreads, and substituting in the expression for $S^3 - kT^2 = 0$ we find the multiplicity of the root $\lambda = 0$. Thus we find.

*On $F_8^4$, $F_7^{24}$ and $F_6^{42}$ are simple, $F_5^{15}$ and $F_4^{24}$ are 2-fold, and $F_2^9$ is 3-fold.*

*On $F_8^6$, $F_7^{24}$ is simple, $F_6^{42}$ and $F_5^{15}$ are 2-fold, $F_4^{24}$ is 3-fold, and $F_2^9$ is 4-fold.*

*On each $F_8^{12}$, $F_7^{24}$ is 2-fold, $F_6^{42}$ is 3-fold, $F_5^{15}$ is 4-fold, $F_4^{24}$ is 6-fold, and $F_2^9$ is 8-fold.*

*On $V_8^{12}$, $F_7^{24}$ and $V_7^{21}$ are 2-fold, $F_6^{42}$ and $V_6^{15}$ are 3-fold, $F_5^{15}$ is 4-fold, $F_4^{24}$ is 6-fold, and $F_2^9$ is 8-fold.*

The multiplicity of any spread on $F_7^{24}$ is equal to the product of its multiplicities on $F_8^6$ and $F_8^4$ since $F_7^{24}$ is the complete intersection of these two hypersurfaces. Hence

*On $F_7^{24}$, $F_6^{42}$ is 2-fold, $F_5^{15}$ is 4-fold, $F_4^{24}$ is 6-fold, and $F_2^9$ is 12-fold.*

An arbitrary plane $\alpha$ in $S_9$ cuts out on $V_8^{12}$ a plane curve $K$ of order 12 whose points are in $(1-1)$ correspondence with the points of the Jacobian

$J$ of the net of cubics in $\pi$ corresponding to $\alpha$. $K$ is therefore of the same genus as $J$. If we take $\alpha$ on a point $P$ of any of the spreads which is of multiplicities $\mu$ and $\nu$ on $V_8^{12}$ and $F_7^{24}$ respectively, $K$ will have a $\mu$-fold point at $P$ and $24-\nu$ double-points on $F_7^{24}$. If the genus of the corresponding Jacobian $J$ is $p$, $K$ will have $\gamma = 11 \cdot 10/2 - p - \mu(\mu-1)/2 - 24 + \nu$ double points on $V_7^{21}$. $21-\gamma$ is the multiplicity of the spread on $V_7^{21}$. Thus we find:

*On $V_7^{21}$, $V_6^{15}$ is 3-fold, $F_6^{42}$ and $F_5^{15}$ are simple.*

If in the linear system $\infty^4$ of cubics (1.2), $\phi_1$ is composed of a double line and a simple line, say $\phi_1 = x_1^2 x_2$ then the elements of the first column in (1.4) are $2x_2$, 0, 0, $2x_1$, 0, 0 and the order of the matrix is still 15. If, however, $\phi_1$ is composed of a triple line, say $\phi_1 = x_1^3$, then the elements of the first column in the above matrix are $6x_2$, 0, 0, 0, 0, 0, and the order of this matrix is equal to the order of the matrix of 5 rows and 4 columns obtained by deleting the first row and column in the above. Using Salmon's formula (1.5) we find its order to be 10. Hence

*On $F_5^{15}$, $F_4^{24}$ is simple, $F_2^9$ is 5-fold.*

In order to obtain the multiplicity of $F_2^9$ on $F_4^{24}$, we take in $\pi$ a linear system $\infty^5$ of cubics which contains a cubic degenerating into a triple line $a^3$, and establish on a line $l$ the same [30, 30] correspondence as in j1. The 60 coincidences in this case are accounted for as follows:

(1) The locus of triple points of the system breaks up into the line $a$ and a quintic $C_5$. The 5 intersections of $l$ and $C_5$ give $5 \cdot 6 = 30$ coincidences.

(2) The intersection of $l$ and $a$ absorbs 10 coincidences since $F_2^9$ is 5-fold on $F_5^{15}$ and $a^3$ absorbs 5 cubics with a triple point of the system $\infty^4$ thru $(al)$.

(3) The remaining $60 - 30 - 10 = 20$ coincidences are due to the number of cubics composed of a double line and a simple line in the system. Therefore $F_2^9$ *is 4-fold on* $F_4^{24}$.

3. *Linear spaces on the spreads.* To a linear system $\infty^k$ of cubics all of which have the same singularity $\alpha$ corresponds evidently a linear space $S_k$ on the spread of cubics $\alpha$. The majority of the spreads contain linear spaces. In certain cases they contain one or more rulings, i. e. systems of linear spaces such that through each point of the spread there passes *one* linear space of the system.

Chisini * shows that there exist $\infty^8$ pencils of cubics of equal generic modulus, $\infty^9$ pencils of harmonic cubics, and $\infty^{11}$ pencils of equianharmonic cubics in the plane. We have then on the corresponding hypersurfaces $\infty^8$ lines on each of the $\infty^1$ $F_8^{12}$'s, $\infty^9$ lines on $F_8^6$, and $\infty^{11}$ lines on $F_8^4$.

In a recent paper † and in a note ‡ which will be published in a later issue of this Journal, I show that every pencil of equianharmonic cubics is contained in a net of such cubics through 6 of the 9 base-points of the pencil, and that the 9 base-points divide into 3 triples such that through any 2 triples there passes a net of equianharmonic cubics. Hence the $\infty^{11}$ lines of $F_8^4$ lie on $\infty^9$ planes in $F_8^4$ such that there are three planes on every line.

A characteristic property of an equianharmonic cubic is that it is left invariant by a cyclic homology of period 3, the axis of which is a flex-line of the cubic and the center is the point of intersection of the three flex tangents at the three flexes on the axis. For a given homology in the plane there is a linear system $\infty^4$ of equianharmonic cubics which are invariant for it (if the axis be $x_1 = 0$ and the center $(1, 0, 0)$ then the linear system is $x_1^3 + \phi_3(x_1x_2) = 0$, $\phi_3$ being a binary cubic form). Since there are $\infty^4$ such homologies in the plane there are $\infty^4$ $S_4$'s on $F_8^4$. On each point of $F_8^4$ there are 9 $S_4$'s since each equianharmonic cubic admits 9 homologies into itself.

In determining the linear spaces on the remaining spreads we use two well known theorems of Bertini § which state that if the general curve of a linear system is irreducible, it can have no variable multiple points, and if the general curve is reducible then either all the curves of the system have a fixed part or each is made up of $k$ variable members of a fixed pencil.

The $V_8^{12}$ of nodal cubics is ruled by $\infty^2$ $S_6$'s corresponding to the linear systems $\infty^6$ of cubics having a double-point at a given point of the plane. Each $S_6$ cuts $F_2^9$ in a cubic curve $\Gamma_3$ corresponding to all the cubics composed of the lines on the given point each counted three times. It has $\infty^1$ planes on $F_4^{24}$, each plane corresponding to the net of cubics composed of a fixed line on the given point counted twice and an arbitrary line of the plane.

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* O. Chisini, "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo* (1916).

† J. Yerushalmy, "Construction of Pencils of Equianharmonic Cubics," *American Journal of Mathematics*, Vol. 53 (1931), pp. 319-332.

‡ "On the Configuration of the 9 Base-Points of a Pencil of Equianharmonic Cubics."

§ E. Bertini, "Sui sistemi lineari," *Rendiconti dell' Istituto Lombardo* (2), Vol. 15 (1882), pp. 24-28.

It cuts $F_5^{15}$ in an $S_3$ corresponding to all the cubics having a triple-point at the given point. The cubic $\Gamma_3$ of $F_2^9$ is in this $S_3$.

The $F_7^{24}$ of cuspidal cubics is ruled by $\infty^3$ $S_4$'s. Each of these corresponds to a system $\infty^4$ of cubics having a cusp at a given point of the plane with a fixed cusp tangent. These $S_4$'s are evidently contained in the $S_6$'s of $V_8^{12}$.

The $V_7^{21}$ of cubics which degenerate into a conic and a line is ruled in two ways. First by $\infty^2$ $S_5$'s each corresponding to the cubics which degenerate into a fixed line and an arbitrary conic. Every $S_5$ has a point in common with $F_2^9$ (the fixed line counted 3 times) and a surface of Veronese ($F_2^4$) in common with $F_4^{24}$ (corresponding to the cubics composed of the fixed line and an arbitrary double line). There are also on $V_7^{21}$ $\infty^5$ planes $S_2$ corresponding to nets of cubics composed of a fixed conic and an arbitrary line of the plane. A generic $S_5$ meets a generic plane in a point (the point corresponding to the cubic composed of the fixed line of the $S_5$ and the fixed conic of the $S_2$) they therefore belong to an $S_7$ (the $\infty^7$ of cubics thru the 2 common points of the fixed line and the fixed conic).

The $F_6^{42}$ is ruled by $\infty^3$ $S_3$'s corresponding to the webs of cubics which degenerate into a fixed line and a web of conics tangent to the line at a given point. Each is contained in an $S_5$ of $V_7^{21}$.

On $V_6^{15}$ there are two kinds of planes. There are $\infty^4$ planes $S_2$ corresponding to the nets of cubics composed of 2 fixed lines and an arbitrary line of the plane. There are also $\infty^4$ planes $\bar{S}_2$ corresponding to the nets of cubics composed of a fixed line and two lines varying in a pencil. On each point of $V_6^{15}$ there are 3 planes $S_2$ and 3 planes $\bar{S}_2$.

The $F_5^{15}$ is ruled by $\infty^2$ $S_3$'s corresponding to the webs having a triple-point at a given point of the plane. They are contained in the $S_6$'s of $V_8^{12}$ and cut out the cubic curves $\Gamma_3$ on $F_2^9$.

The $F_4^{24}$ is ruled by $\infty^2$ planes corresponding to the nets of cubics having a fixed double line and a variable line. Each plane has a point in common with $F_2^9$.

The $F_4^{24}$ contains also $\infty^2$ surfaces of Veronese ($F_2^4$) corresponding to the cubics having a fixed line and a variable double line of the plane. Each $F_2^4$ is contained in an $S_5$ on $V_7^{21}$, and cuts $F_2^9$ in a point.

The $F_2^9$ contains no linear spaces.

4. *The* $F_2^9$. We have seen that there are $\infty^2$ cubics $\Gamma_3$ on $F_2^9$. The points of such a cubic correspond to the cubics composed of a triple-line varying in a pencil. The $\Gamma_3$'s are ordinary space cubics belonging to the $\infty^2$ $S_3$'s of $F_5^{15}$.

The cubics composed of the tangents to a given conic $C_2$ in $\pi$ counted 3 times map on the points of a curve $\Gamma_6$ of order 6. Such a curve is contained in an $S_6$ corresponding to the linear system $\infty^6$ of cubics apolar to the net of curves of class 3 composed of $C_2$ and an arbitrary point of the plane. It is a rational norm-curve in $S_6$. There are, evidently, $\infty^5$ curves $\Gamma_6$ on $F_2^9$.

The $\infty^9$ hyperplanes $S_8$ in $S_9$ cut out on $F_2^9$ $\infty^9$ curves $\Gamma_9$ of order 9, each an elliptic norm-curve in its $S_8$.

It is easily verified by referring to $\pi$ that:

Two cubics $\Gamma_3$ intersect in one point. There is a single $\Gamma_3$ on any two points of $F_2^9$ and $\infty^1$ on one point.

Two curves $\Gamma_6$ intersect in 4 points. On any 5 points of $F_2^9$ there is a single $\Gamma_6$ and $\infty^1$, $\infty^2$, $\infty^3$, $\infty^4$ curves $\Gamma_9$ on 4, 3, 2, 1 points respectively.

Two curves $\Gamma_9$ intersect in 9 points (the 9 points in which the $S_7$, common to the two hyperplanes which cut out the two curves, cuts $F_2^9$). There are 1, $\infty^1$, $\infty^2, \cdots \infty^8$ curves $\Gamma_9$ on respectively 9, 8, $7, \cdots 1$ points of $F_2^9$.

A cubic $\Gamma_3$ and a sextic $\Gamma_6$ intersect in two points (the 2 points corresponding to the two tangents drawn from the center of the pencil, corresponding to $\Gamma_3$, to the conic $C_2$ counted 3 times).

A cubic $\Gamma_3$ and a curve $\Gamma_9$ have 3 points in common (the 3 points in which the plane common to $S_3$ and $S_8$ cuts $\Gamma_3$).

A curve $\Gamma_6$ and a curve $\Gamma_9$ intersect in 6 points (the 6 points in which the $S_5$ common to $S_6$ and $S_8$ cuts $\Gamma_6$).

Take a point $P$ of $F_2^9$ and the corresponding triple line $a^3$ in $\pi$. A cubic $\Gamma_3$ on $P$ corresponds to the lines of a pencil of center $Q$ on $a$ each counted 3 times. To the pencil of cubics composed of the line $a$ counted twice and an arbitrary line of the pencil $\{Q\}$ corresponds a line in $S_9$ which is tangent to $\Gamma_3$ (and also to $F_2^9$) at $P$. The $\infty^1$ tangent lines to the $\infty^1$ cubics $\Gamma_3$ at $P$ are in a plane which is the tangent plane to $F_2^9$ at $P$. Hence

*The tangent plane $T_2$ to $F_2^9$ at a point $P$ (the image of a triple line $a^3$), corresponds to the net of cubics composed of the line $a$ counted twice and an arbitrary line of the plane.*

*The $\infty^2$ tangent planes to $F_2^9$ generate the $F_4^{24}$.*

An hyperplane $S_8$ which contains a tangent plane $T_2$ is a tangent hyperplane to $F_2^9$ at the point $P$ in which $T_2$ touches $F_2^9$. We denote such an hyperplane by $R_8$. The section of $R_8$ on $F_2^9$ is a curve $\Gamma_9$ having a double point at the point of contact. It is the generic artional nodal 9-ic curve in $R_8$.

An hyperplane containing two tangent planes $T_2$ is a double tangent hyperplane. It contains the $S_3$ defined by the intersection of the two fixed double lines $a_1^3$, $a_2^3$ in $\pi$ corresponding to the 2 planes $T_2^1$, $T_2^2$ (since it contains the 4 linearly independent points $a_1^3$, $a_2^3$, $a_1^2a_2$, $a_1a_2^2$) and hence contains the cubic $\Gamma_3$ on $a_1^3$, $a_2^3$. The section of such an hyperplane degenerates into the cubic $\Gamma_3$ and a sextic $\Gamma_6$.

An hyperplane containing 3 tangent planes $T_2$ is a triple tangent hyperplane. It contains the 3 $S_3$'s defined by the 3 points of intersection of the 3 fixed lines in $\pi$ and its section degenerates into 3 cubics $\Gamma_3$.

We proceed to determine in $\pi$ the linear systems of cubics which map into the various tangent hyperplanes of $F_2^9$ in $S_9$.

Consider the linear system $\infty^5$ of cubics in $\pi$ passing through 2 points $P$ and $Q$ and tangent there to two lines $p$ and $q$. On the line $PQ$ take two other points $R$, $S$ and consider the linear system $\infty^5$ of cubics having a double point at $R$ and passing simply through $S$. These two systems have in common the net of cubics composed of the line $PQ$ counted twice and an arbitrary line of the plane. They are therefore contained in a linear system $\infty^8$ of cubics. The corresponding $R_8$ in $S_9$ contains the $T_2$ corresponding to the common net and is therefore tangent to $F_2^9$ at the point $\overline{PQ^3}$. There are $\infty^8$ such hyperplanes in $S_9$ (we can pick $P$, $Q$, $R$, $S$, $p$, $q$ in $\infty^8$ ways). There are $\infty^6$ such hyperplanes tangent to $F_2^9$ at a given point.

Take in $\pi$ 3 points $R$, $S$, $T$ on a line $a$ and another point $P$ not on $a$. Consider the two linear systems $\infty^5$ of cubics, the one having $R$ as a double base-point and $S$ as a simple base-point, the other having $P$ and $T$ as double and simple base-points respectively. These two systems have in common the net of cubics which degenerates into the line $a$ and the net of conics on $PRS$, and belong therefore to a linear system $\infty^8$ of cubics. The hyperplane $R_8$ corresponding to this linear system contains the two $S_5$'s on $V_8^{12}$ corresponding to the two linear systems $\infty^5$ of cubics. It contains therefore the two tangent planes $T_2$ to $F_2^9$ at the points $a^3$ and $\overline{PT^3}$, and is a double tangent hyperplane. Its section degenerates into a cubic $\Gamma_3$ and a sextic $\Gamma_6$. There are obviously $\infty^7$ such hyperplanes in $S_9$, $\infty^5$ tangent at a given point of $F_2^9$, and $\infty^3$ tangent at two given points of $F_2^9$.

Consider three webs of cubics having a triple point. The corresponding three dimensional spaces $S_3^{(1)}$, $S_3^{(2)}$, $S_3^{(3)}$ on $F_5^{15}$ intersect by twos in three points $A$, $B$, $C$ of $F_2^9$. The $S_6(S_3^{(1)}S_3^{(2)})$ and $S_3^{(3)}$ have the two points $B$, $C$ in common. They have, hence, the line $BC$ in common and belong to an $R_8$. This hyperplane contains the three tangent planes $T_2$ at the points $A$, $B$, $C$ and is a triple tangent hyperplane. The 9-ic $\Gamma_9$ which it cuts out

on $F_2^9$ is composed of the three cubics $\Gamma_3$ cut out on $F_2^9$ by $S_3^{(1)}$, $S_3^{(2)}$, $S_3^{(3)}$. There are $\infty^6$ such $R_8$ in $S_9$, $\infty^5$ tangent to a given point of $F_2^9$, $\infty^2$ tangent to any 2 points, and a single $R_8$ tangent to 3 given points of $F_2^9$.

If the above 3 webs are such that the three fixed triple points are on a line, then the corresponding spaces $S_3^{(1)}$ $S_3^{(2)}$ $S_3^{(3)}$ meet in a point $A$ of $F_2^9$. The $S_6(S_3^{(1)}S_3^{(2)})$ contains the tangent lines to $\Gamma_3^{(1)}$ and $\Gamma_3^{(2)}$, it contains therefore the tangent plane $T_2$ to $F_2^9$ at $A$, and since the tangent line to $\Gamma_3^{(3)}$ at $A$ is also in $T_2$, the three spaces $S_3^{(1)}$ $S_3^{(2)}$ $S_3^{(3)}$ have a line in common and are contained in an $R_8$. This $R_8$ is tangent to $F_2^9$ at $A$. The 9-ic which it cuts out on $F_2^9$ has a triple point at $A$ and is composed of the three cubics $\Gamma_3^{(1)}$ $\Gamma_3^{(2)}$ $\Gamma_3^{(3)}$. There are $\infty^5$ such $R_8$ in $S_9$, and $\infty^3$ tangent at a given point of $F_2^9$.

The linear system $\infty^8$ of cubics apolar to a given curve of class 3 degenerating into a double-point $P$ and a simple point $Q$ contains the cubics composed of the lines of the pencils $\{P\}$ and $\{Q\}$ each counted 3 times. It also contains all the cubics which break up into a line of the pencil $\{P\}$ counted twice and an arbitrary line of $\pi$. The $R_8$ corresponding to this system cuts out on $F_2^9$ two cubics $\Gamma_3^{(1)}$ and $\Gamma_3^{(2)}$ and contains all the tangent planes $T_2$ at the points of one of these cubics say $\Gamma_3^{(1)}$ and is therefore tangent to $F_2^9$ along this cubic. Its section with $F_2^9$ is composed of $\Gamma_3^{(1)}$ counted twice and $\Gamma_3^{(2)}$. There are $\infty^4$ such $R_8$ and $\infty^2$ tangent at any point of $F_2^9$.

The linear system $\infty^8$ of cubics passing through a given point $P$ of $\pi$ contains the cubics which degenerate into the lines of the pencil $\{P\}$ each counted 3 times. It contains also the cubics composed of a line of $\{P\}$ counted twice and an arbitrary ilne of $\pi$. The $R_8$ corresponding to this system cuts out on $F_2^9$ a cubic $\Gamma_3$ and contains the tangent planes $T_2$ to $F_2^9$ at all the points of $\Gamma_3$. It is therefore a triple tangent hyperplane to $F_2^9$ along this cubic and its section is made up of $\Gamma_3$ counted three times. There are $\infty^2$ such hyperplanes in $S_9$, and $\infty^1$ tangent at any point of $F_2^9$.

If we consider in the plane $\pi$ a correlation between its points and its lines, then the system $\infty^9$ of curves of order 3 in it is transformed into the system $\infty^9$ of curves of class 3. $F_2^9$ is then the image of all the curves in $\pi$ composed of triple points, and there is a $(1-1)$ correspondence between the points of $F_2^9$ and the points of the plane. If we consider now an hyperplane $S_8$ which corresponds to a linear system $\infty^8$ of curves of class 3, as representing the cubic curve (of order 3) apolar to that system, then we have established besides the $(1-1)$ correspondence between the curves of class 3 of $\pi$ and the points of $S_9$, also a $(1-1)$ correspondence between the cubic

curves of $\pi$ and the hyperplanes of $S_9$. The apolarity condition of a system $\infty^h$ of curves of class 3 and the system $\infty^{8-h}$ of curves of order 3 is expressed by the incidence condition of the corresponding linear spaces. $F_2^9$ appears then as the map of the plane $\pi$ by means of the system $\infty^9$ of all its cubics. The lines of $\pi$ go by this mapping into the $\infty^2$ cubics $\Gamma_3$, the conics of $\pi$, into the curves $\Gamma_6$ and the cubics of $\pi$ go into the hyperplane sections $\Gamma_9$. In general any curve $n$ of $\pi$ is mapped into a curve of order $3n$. The sections of $F_2^9$ by the various tangent hyperplanes $R_8$ correspond to the cubics having one or more double-points. In general a curve $\Gamma_9$ corresponding to a cubic curve with any singularity $\alpha$ is cut out by the hyperplane corresponding to the linear system $\infty^8$ of curves of class 3 apolar to that cubic.

The $F_2^9$ was first met by Del Pezzo * in studying the surfaces of order $n$ immersed in a space of $n$ dimensions. He proves that such a surface is either a ruled surface or $F_2^9$ or a projection of it on a space of lower dimensions. $F_2^9$ is not a projection of a surface of the same order of a space of higher dimensions. It is a norm-surface. Del Pezzo studies the projection of $F_2^9$ on spaces of lower dimensions and in particular its projection in $S_3$. He obtains the seven surfaces of orders 9, 8, $\cdots$ 3 depending on the number of points that the center of projection has in common with $F_2^9$. If we project $F_2^9$ from 6 points on it we obtain in $S_3$ the general cubic surface. The six centers of projection go into a sextuple of lines on the cubic surface, and since there are no lines on the cubic surface that are skew to the six lines of the sextuple it follows that there are no lines on $F_2^9$. Any of the 27 lines of the cubic surface which meets $k$ lines of the sextuple represents a rational curve of order $k+1$ of $F_2^9$ which passes through the corresponding $k$ points. The surfaces $\phi^3$, $\phi^4$, $\cdots$ $\phi^9$ in $S_3$ which are the projections of $F_2^9$ from $r$ points on it ($r = 6, 5, \cdots 1, 0$) are the map of the plane by means of a linear system $\infty^3$ of cubics with $r$ base-points.

Castelnuovo † shows that every non-ruled surface in $S_3$ whose plane sections are elliptic curves is rational and of order $n \leqq 9$ and is the projection of $F_2^9$.

5. *Extesnion to plane curves of order n.* Most of the properties of $S_9$ can be extended to the $S_N$ $[N = n(n+3)/2]$ whose points represent the plane curves of order $n$. By the same reasoning as for $S_9$ we can state that:

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* P. Del-Pezzo, "Sulle superficie dell' $n$—*mo* ordine immerse nelle spazio di $n$ dimensioni," *Rendiconti del Circolo Matematico di Palermo*, Vol. 1, p. 241.

† G. Castelnuovo, "Sulle superficie algebriche le cui sezioni piane sono curve ellitiche," *Rendiconti dell' Accademia dei Lincei* (1894), p. 59.

*The spread of nodal n-ics is a* $V^{3(n-1)^2}_{N-1}$.

*The spread of n-ics with 2 double points is a* $V^{\alpha}_{N-2}$

$$\alpha = \tfrac{3}{2}(n-1)(n-2)(3n^2-3n-11).$$

*The spread of cuspidal n-ics is an* $F^{12(n-1)(n-2)}_{N-2}$.

*The spread of n-ics having a tacnode is an* $F^{\beta}_{n-3}$

$$\beta = 2(25n^2-96n+84).$$

*The spread of n-ics with 3 double points is a* $V^{\gamma}_{N-3}$

$$\gamma = \tfrac{1}{6}\,[(3n^2-6n+4)^3 - 3(3n^2-6n+4)(45n^2-123n+82) + 1542n^2 - 5151n + 4070].$$

*The spread of n-ics having a triple-point is an* $F^{15(n-2)^2}_{N-4}$, for the elements of the matrix (1.4) are in this case of order $n-2$ and applying Salmon's formula its order is $15(n-2)^2$.

In general *the spread of n-ics having a k-fold point is an* $F^{3\binom{k+2}{4}(n-k+1)^2}_{N+2-[k(k+1)/2]}$.

For the condition for a $k$-fold point is that the $(k-1)$-th derivatives vanish. The order of the spread is therefore equal to the order of the matrix of $k(k+1)/2$ rows and $k(k+1)/2-1$ columns

$$\begin{pmatrix} \dfrac{\partial^{(k-1)}\phi_1}{\partial x_1^{(k-1)}} & \dfrac{\partial^{(k-1)}\phi_2}{\partial x_1^{(k-1)}} & \cdots & \dfrac{\partial^{(k-1)}\phi_{[(k-1)(k+2)/2]}}{\partial x_1^{(k-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{(k-1)}\phi_1}{\partial x_1^{i}\partial x_2^{j}\partial x_3^{l}} & \cdots & \cdots & \dfrac{\partial^{(k-1)}\phi_{[(k-1)(k+2)/2]}}{\partial x_1^{i}\partial x_2^{j}\partial x_3^{l}} \end{pmatrix}$$
$$(i+j+l=k),$$

whose elements are of order $n-k+1$. Salmon's formula gives for its order the number $3\binom{k+2}{4}(n-k+1)^2$.

For $k=n$ we obtain

*The spread of n-ics with an n-fold point is an* $F^{3\binom{n+2}{4}}_{n+2}$.

The order of the spread of $n+1$ dimensions of $n$-ics with an $n$-fold point composed of a double line and $n-2$ simple lines is obtained by setting up the same correspondence as in the case of the $F_4^{24}$ (j1, § 1). We have in this case on the line $l$ a $\left[3(n-1)\binom{n+2}{4}_1, 3(n-1)\binom{n+2}{4}\right]$ correspondence with $6(n-1)\binom{n+2}{4}$ coincidences. The $n(n+1)/2$ intersection of $l$ with

locus of $n$-fold points of the system $\infty^{(n-1)(n+2)/2}$ of $n$-ics given by the determinant

$$\left| \frac{\partial^{(n-1)}\phi_a}{\partial x_1{}^i \partial x_2{}^j \partial x_3{}^l} \right| = 0$$

account for $n(n-1) \cdot n(n+1)/2$ coincidences leaving

$$6(n-1)\binom{n+2}{4} - n^2(n+1)(n-1)/2 = 6(n+1)\binom{n+1}{4}$$

coincidences which arise from the same number of $n$-ics with an $n$-fold point, in the system which degenerate into a double line and $n-2$ simple lines. Hence

*The spread of n-ics with an n-fold point composed of a double line and* $n-2$ *simple lines is an* $F_{n+1}^{6(n+1)\binom{n+1}{4}}$.

In order to obtain the order of the spread $F_4$ of 4 dimensions of $n$-ics which degenerate into an $(n-1)$-fold line and a simple line we take an arbitrary linear space $S_{N-4}$ in $S_N$. On $S_{N-4}$ there is a web of hyperplanes $S_{N-1}$. They cut out on $F_2{}^{n^2}$ (the surface of $n$-fold lines) a web of curves of order $n^2$ which correspond to a web of curves of order $n$ in the plane. At an intersection of $S_{N-4}$ with the $F_4$ there is for $n > 2$ one tangent plane $T_2$ to $F_2{}^{n^2}$. This $T_2$ and $S_{N-4}$ define an $S_{N-2}$ (since they intersect in a point). On this $S_{N-2}$ there is a pencil of hyperplanes of the web. Each hyperplane of this pencil is a tangent hyperplane to $F_2{}^{n^2}$ at the point of contact of $T_2$. The number of intersections of $S_{N-4}$ with $F_4$ (i. e. the order of $F_4$) is therefore equal to the number of pencils of curves $n$ having a double base-point which is contained in an arbitrary linear system $\infty^3$ of $n$-ics

$$\lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_4\phi_4 = 0.$$

This number is evidently given by the order of the matrix

$$\begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_2}{\partial x_1} \cdots \frac{\partial\phi_4}{\partial x_1} \\ \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_2}{\partial x_2} \cdots \frac{\partial\phi_4}{\partial x_2} \\ \frac{\partial\phi_1}{\partial x_3} & \frac{\partial\phi_2}{\partial x_3} \cdots \frac{\partial\phi_4}{\partial x_3} \end{pmatrix}$$

of 3 rows and 4 columns whose elements are of order $n-1$. By Salmon's formula this matrix is of order $6(n-1)^2$. Hence

*The spread of n-ics composed of an $(n-1)$-fold line and a simple line is an $F_4^{6(n-1)^2}$.*

This is a good check on the correspondence by which we obtained the order of $F_4^{24}$.

*The spread of n-ics composed of a line counted n times is an $F_2^{n^2}$.*

The points of this surface are in $(1-1)$ correspondence with the point of the plane. It is the map of the plane by means of all the curves of order $n$. The hyperplane sections are norm-curves of order $n$ and of genus $(n-1)(n-2)/2$. Corresponding to the lines of the plane there are on $F_2^{n^2}$ $\infty^2$ curves $\Gamma_n$ of order $n$. These are rational norm-curves and belong to an $S_n$ on $F^{3\binom{n+2}{4}}_{(n+2)}$. On each point $P$ of $F_2^{n^2}$ there are $\infty^1$ curves $\Gamma_n$. The tangent lines to these at $P$ are in the tangent plane to $F_2^{n^2}$ at this point, which is, in fact, the plane corresponding to the net of $n$-ics in the plane composed of a fixed line $a$ $(a^3 \equiv P)$ counted $(n-1)$ times and an arbitrary line of the plane. The $\infty^2$ tangent planes to $F_2^{n^2}$ at all its points generate the $F_4^{6(n-1)^2}$.

The only curves on $F_2^{n^2}$ are those whose order is a multiple of $n$, $(\mu n)$, and correspond to the curves of order $\mu$ in the plane.

The $F_4^{6(n-1)^2}$ contains $\infty^2$ surfaces $F_2^{(n-1)^2}$ corresponding to the $n$-ics in the plane composed of a fixed line $a$ and an arbitrary line counted $n-1$ times. Each such $F_2^{(n-1)^2}$ is contained in an $S_{(n-1)(n+2)/2}$ corresponding to the linear system $\infty^{(n-1)(n+2)/2}$ of $n$-ics composed of the fixed line $a$ and an arbitrary curve of order $n-1$.

The projection of $F_2^{n^2}$ on $S_3$ from $r$ points on it $[r = 0, 1, 2, \cdots, (n^2+3n-6)/2]$ is a surface of order $n^2 - r$ which is the map of the plane by means of a system $\infty^3$ of curves of order $n$ with $r$ base points.

# ON NON-SINGULAR BOUNDED MATRICES.

By AUREL WINTNER.

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We understand by $A = \| a_{ik} \|$ an infinite matrix which is bounded in the sense of Hilbert and denote the matrix $\| \bar{a}_{ki} \|$ by $A^*$, so that $A^*$ represents the transposed matrix of the conjugate complex elements. If there exists a bounded matrix $B$ for which both products $AB$, $BA$ are equal to the unity matrix $E$ then $B$ is, according to Toeplitz, uniquely determined and may therefore be denoted by $A^{-1}$. We call the bounded matrix $A$ non-singular if the bounded matrix $A^{-1}$ exists. The product of bounded matrices is a bounded matrix. The bounded matrix is Hermitean if $A^* = A$ and unitary if $A^* = A^{-1}$. The unitary matrices form a group of bounded matrices. A bounded matrix is called positive definite if it is Hermitean, non-singular and such that the corresponding Hermitean form cannot take a negative value. The reciprocal matrix of a bounded positive definite matrix always exists and is also positive definite. If $V$ is a unitary and $Q$ a positive definite bounded matrix then $VQV^{-1}$ is also a positive definite bounded matrix.—We shall demonstrate the following theorem:

*For any non-singular bounded matrix $A$ there exists exactly one positive definite matrix $P$ and exactly one unitary matrix $U$ for which $A = PU$.*

We can call $P$ the modulus and $U$ the argument of $A$. Since the product of a positive definite bounded matrix and a unitary matrix is always a non-singular bounded matrix, our "polar" representation $A = PU$ yields a one-to-one correspondence between the non-singular bounded matrices $A$ and the *pairs* of bounded positive definite and unitary matrices $P$, $U$. We infer that $PU$ is not always equal to $UP$. Since $VPV^{-1}$ is positive definite and $VUV^{-1}$ unitary for any unitary matrix $V$ it follows from the identity

$$VAV^{-1} = VPUV^{-1} = VPV^{-1}VUV^{-1}$$

and from the uniqueness of the polar components of a non-singular bounded matrix that $VPV^{-1}$ and $VUV^{-1}$ are the polar components of $VAV^{-1}$. *The unitary invariants of the polar components $P$, $U$ of $A$ are, therefore, unitary invariants of $A$ itself.*

The demonstration is based on the fact † that every bounded positive definite matrix possesses exactly one bounded positive definite "square root." In other words there exists for any bounded positive definite matrix $Q$ a bounded positive definite matrix $P$ for which $Q = P^2$; furthermore, we have

$$P_1 = P_2 \quad \text{if} \quad P_1^2 = P_2^2, \tag{1}$$

provided $P_1$ and $P_2$ are bounded and positive definite.

Since $(CD)^* = D^*C^*$, we have $(AA^*)^* = AA^*$, i.e. $AA^*$ is always a Hermitean matrix. Furthermore, the Hermitean form belonging to $AA^*$ can be written in the form $\sum_i |\sum_k a_{ik}x_k|^2 \geqq 0$. Finally, $AA^*$ possesses the bounded reciprocal $A^{-1*}A^{-1}$ provided $A^{-1}$ exists. Therefore $AA^*$ is for any non-singular matrix $A$ a positive definite bounded matrix.

Let $A$ be any non-singular bounded matrix. Let $P$ denote the positive definite square-root of $AA^*$:

$$P^2 = AA^*. \tag{2}$$

We define a bounded matrix $U$ as follows:

$$U = P^{-1}A. \tag{3}$$

We then have

$$U^* = A^*P^{-1}, \tag{3'}$$

inasmuch as the reciprocal $P^{-1}$ of the Hermitean matrix $P$ is a Hermitean matrix. From (2), (3) and (3') it follows

$$UU^* = P^{-1}AA^*P^{-1} = P^{-1}P^2P^{-1} = E,$$
$$U^*U = A^*P^{-2}A = A^*(AA^*)^{-1}A = E,$$

i.e. the matrix (3) is a unitary matrix and (3) therefore yields the polar representation $A = PU$.

In order to show that this polar representation of $A$ is unique, suppose that

$$A = P_1U_1 = P_2U_2. \tag{4}$$

We then have $A^* = U_1^{-1}P_1$, $A^* = U_2^{-1}P_2$ and therefore $AA^* = P_1^2 = P_2^2$, or, by virtue of (1), simply $P_1 = P_2$. Since $P_1 = P_2$ is by supposition positive definite and therefore non-singular equation (4) yields the further condition $U_1 = U_2$ which finishes the demonstration for the uniqueness.

It may be mentioned that the matrix $AA^*$ corresponds to the *dilatation*

† Cf. A. Wintner, "Zur Theorie der beschränkten Bilinearformen," *Mathematische Zeitschrift*, Vol. 30 (1929), p. 267 (below).

*ellipsoid* of the "deformation" defined by $A$. The unique resolution of $A$ into the polar components $P$ and $U$ corresponds to the resolution of the *deformation* $A$ into a *dilatation* $P$ and a *rotation* $U$ which are not necessarily commutable. However there also exists a unique representation $A = WR$ where $W$ is a unitary matrix and $R$ a positive definite bounded matrix. To show this we need only write $A = PU = U(U^{-1}PU)$ where $U = W$ is a unitary matrix and $U^{-1}PU = R$ a positive definite bounded matrix. If $A$ is a real matrix the positive definite square root $P$ of $AA^*$ is also real † as is the matrix $P^{-1}A = U$.

It is not difficult to determine the non-singular bounded matrices for which the both polar components $P$, $U$ are commutable,‡

$$A = PU = UP \qquad (\text{or } P = R,\ U = W). \tag{5}$$

Equation (5) is then and only then fulfilled if $A = PU$ is a so-called normal matrix:

$$AA^* = A^*A. \tag{6}$$

Since (5) can be written in the form

$$A^* = U^{-1}P = PU^{-1} \tag{5}'$$

equation (6) follows obviously from (5). Conversely (5) follows from (6). In fact, on introducing $A = PU$, $A^* = U^{-1}P$ in (6) we obtain $P^2 = U^{-1}P^2U$ or $(UPU^{-1})^2 = P^2$, that is, by virtue of (1), simply $UPU^{-1} = P$ which is the same as (5). *The non-singular bounded matrix $A$ is, therefore, then and only then normal if its polar components are commutable.*

In the following we give some additional applications of the positive definite square root.

If $G_1, G_2, \cdots, G_r$ denotes a finite group of bounded matrices then $G_mG_1, G_mG_2, \cdots, G_mG_r$ represents, for any fixed value of $m$, only a permutation of $G_1, G_2, \cdots, G_r$, so that

$$\sum_{n=1}^{r} G_nG_n^* = \sum_{n=1}^{r} (G_mG_n)(G_mG_n)^* = \sum_{n=1}^{r} G_mG_nG_n^*G_m^*$$

or

$$\Gamma = G_m\Gamma G_m^* \qquad (m = 1, 2, \cdots, r) \tag{7}$$

where

$$\Gamma = \sum_{n=1}^{r} G_nG_n^*.$$

† Cf. *loc. cit.*, p. 267.

‡ Cf. *loc. cit.*, p. 282.

The matrix $\Gamma$ is the sum of positive definite bounded matrices and is, therefore, bounded and positive definite. On denoting by $\Pi$ the positive definite square root of the positive definite bounded matrix $\Gamma^{-1}$ we have

$$\Gamma^{-1} = \Pi^2, \qquad \Pi = \Pi^*. \tag{8}$$

Equations (7) and (8) yield

$$\begin{aligned}(\Pi G_m \Pi^{-1})(\Pi G_m \Pi^{-1})^* &= (\Pi G_m \Pi^{-1})(\Pi^{-1} G_m^* \Pi) = \Pi G_m \Pi^{-2} G_m^* \Pi \\ &= \Pi G_m \Gamma G_m^* \Pi = \Pi \Gamma \Pi = \Pi \Pi^{-2} \Pi = E\end{aligned}$$

and in an analogous manner $(\Pi G_m \Pi^{-1})^* (\Pi G_m \Pi^{-1}) = E$, i. e. $\Pi G_m \Pi^{-1}$ is a unitary matrix. *There, therefore, exists, for any finite group* $G_1, G_2, \cdots, G_r$ *of bounded matrices, a positive definite bounded matrix* $\Pi$ *in such a manner that the isomorphic group* $\Pi G_1 \Pi^{-1}, \Pi G_2 \Pi^{-1}, \cdots, \Pi G_r \Pi^{-1}$ *contains unitary matrices only.* It may be mentioned that one could apply, just as in the usual demonstration for finite matrices, instead of the positive definite matrix $\Pi$, a recursive ("triangle") matrix which is furnished by the Jacobi-Toeplitz reduction of $\Gamma$.

If $C$ is a bounded cyclic matrix of the order $r$ (i. e. $E = C^r$ and $r$ the least positive integer $p$ for which $C^p = E$) then

$$C, C^2, \cdots, C^{r-1}, C^r$$

represents a finite group. There exists, therefore, a positive definite bounded matrix $\Pi$ for which $\Pi C \Pi^{-1}$ is a unitary matrix $U$. Putting $P = \Pi^{-1}$ we conclude that *there exists, for any bounded matrix* $C$ *for which* $C^r = E$, *a positive definite bounded matrix* $P$ *and a unitary matrix* $U$ *for which* $C = PUP^{-1}$; *furthermore, the spectrum of* $C$ *contains at most* $r$ *numbers and these are always* $r$*-th roots of unity.* In fact, if $\lambda^r \neq 1$, the formula

$$L = (1 - 1/\lambda^r)^{-1} \sum_{n=0}^{r-1} C^n/\lambda^{n+1}$$

defines a bounded matrix for which we have, by virtue of $C^{n+r} = C^n$, obviously $L(\lambda E - C) = E$ and $(\lambda E - C)L = E$; i. e. $(\lambda E - C)^{-1}$ exists provided $\lambda^r \neq 1$.

If $P$ is a positive definite matrix and $U$ a unitary matrix, the matrix $PUP^{-1}$ is not necessarily a unitary matrix (in accordance with the fact that not every matrix belonging to a finite group is a unitary matrix). In a more precise manner $PUP^{-1}$ is only in the trivial case $PUP^{-1} = U$ a unitary matrix. In fact, if $PUP^{-1}$ is a unitary matrix we have $(PUP^{-1})^* = (PUP^{-1})^{-1}$ or $P^{-1}U^{-1}P = PU^{-1}P^{-1}$, i. e. $P^2 = (UPU^{-1})^2$, and, therefore, according to (1), simply $P = UPU^{-1}$ or $PU = UP$, i. e.

$$PUP^{-1} = U. \tag{9}$$

If $T$ is a non-singular bounded matrix and $U_1$ a unitary matrix then $TU_1T^{-1}$ is not necessarily a unitary matrix. However, if $U_1$ and $U_2$ are unitary matrices for which

$$TU_1T^{-1} = U_2, \tag{10}$$

then there exists a *unitary* matrix $W$ for which

$$WU_1W^{-1} = U_2. \tag{11}$$

The demonstration proceeds as follows. On using the polar representation $T = PW$ of the non-singular bounded matrix $T$ equation (10) takes the form

$$PWU_1W^{-1}P^{-1} = U_2, \tag{10'}$$

i. e. the positive definite bounded matrix $P$ transforms the unitary matrix $WU_1W^{-1}$ in a unitary matrix. From (9) it follows, therefore, $PWU_1W^{-1}P^{-1} = WU_1W^{-1}$, or, according to (10'), simply (11). *Similar unitary matrices are accordingly always unitarily equivalent* or all invariants of the unitary matrices are unitary invariants. The same theorem holds, according to a verbal communication of O. Toeplitz, for equivalent Hermitean matrices (this theorem may be shown also with the use of the polar representation of $T$). The complete system of the unitary invariants of a bounded Hermitean matrix is due to the Dissertation of Hellinger. It has been shown,† with the use of a trigonometrical momentum problem,‡ that the treatment of the complete system of the unitary invariants of the unitary matrices may be reduced to the Hermitean problem solved by Hellinger.

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† *Loc. cit.*, p. 274, p. 257.

‡ *Loc. cit.*, p. 269.

# CONGRUENT GRAPHS AND THE CONNECTIVITY OF GRAPHS.*

By HASSLER WHITNEY.

*Introduction.* We give here conditions that two graphs be congruent and some theorems on the connectivity of graphs, and we conclude with some applications to dual graphs. These last theorems might also be proved by topological methods. The definitions and results of a paper by the author on "Non-separable and planar graphs," † will be made use of constantly. We shall refer to this paper as N. For convenience, we shall say two arcs touch if they have a common vertex.

1. *Congruent graphs.* For a definition, see N, 7.‡ In this section we consider no graphs containing 1- or 2-circuits; that is, each arc joins two distinct vertices, and any two vertices are joined by at most a single arc.

THEOREM 1. *Let $G$ and $G'$ be two connected graphs, neither of which consists of three arcs of the form $ab$, $ac$, $ad$. Let there be a 1-1 correspondence between their arcs so that to any two arcs having a common vertex in one graph correspond two arcs having a common vertex in the other. Then $G$ and $G'$ are congruent.*

Case A. *At least one of the graphs, say $G$, is of one of the following forms:*

$$G_a : \alpha(ab), \beta(ac), \gamma(bc), \delta(bd),$$
$$G_b : \alpha(ab), \beta(ac), \gamma(bc), \delta(bd), \epsilon(cd),$$
$$G_c : \alpha(ab), \beta(ac), \gamma(bc), \delta(bd), \epsilon(cd), \zeta(ad).$$

Consider first the graph $G_a$. Let $G'_a$ be the corresponding graph, and let $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ be the corresponding arcs of $G'_a$. As $\alpha$ and $\beta$ touch, $\alpha'$ and $\beta'$ touch. Hence they are of the form $\alpha'(a'b')$, $\beta'(a'c')$. $\delta$ touches $\alpha$ but not $\beta$; hence $\delta'$ touches $\alpha'$ but not $\beta'$; it has therefore $b'$, but neither $a'$ nor $c'$, as end vertex, and is of the form $\delta'(b'd')$. Now $\gamma$ touches $\alpha$, $\beta$ and $\delta$. Hence $\gamma'$ touches $\alpha'$, $\beta'$ and $\delta'$. $\gamma'$ is not of the form $\gamma'(a'b')$, for then $\alpha'(a'b')$ and

* Presented to the American Mathematical Society, February 28, 1931.

† An outline of this paper will be found in the *Proceedings of the National Academy*, Vol. 17 (1931), pp. 125-127. We refer to this outline as P. The full paper has not yet appeared.

‡ P, 3. We formerly used the term *homeomorphic*.

$\gamma'(a'b')$ would form a 2-circuit. It is thus either $\gamma'(b'c')$ or $\gamma'(a'd')$. In the first case, upon dropping primes, $G'_a$ becomes identical with $G_a$. In the second case, we interchange the names $a'$ and $b'$, $c'$ and $d'$, $\beta'$ and $\delta'$. Upon dropping primes, $G'_a$ becomes identical with $G_a$. Thus in either case, $G_a$ and $G'_a$ are congruent.

Consider next the graph $G_b$. We form $G_b$ from $G_a$ by adding the arc $\epsilon(cd)$. $\epsilon$ touches all the preceding arcs but $\alpha$. The same must be true of $\epsilon'$. and hence $\epsilon'$ is of the form $\epsilon'(c'd')$. Just as above, $G_b$ and $G'_b$ are congruent.

Consider finally the graph $G_c$. We form it from $G_b$ by adding the arc $\zeta(ad)$. $\zeta$ touches all the preceding arcs but $\gamma$. The same must be true of $\zeta'$. Hence, if $\gamma'$ is of the form $\gamma'(b'c')$, $\zeta'$ is of the form $\zeta'(a'd')$; if $\gamma'$ is of the form $\gamma'(a'd')$, $\zeta'$ is of the form $\zeta'(b'c')$. In either case, just as before, $G_c$ and $G'_c$ are congruent.

Before proceeding, we shall prove the following

LEMMA. *Let $G$ and $G'$ satisfy the conditions of the theorem. Suppose $G$ contains $G_a$ as a subgraph, and contains also an arc which has just one end vertex in $G_a$. Let $G'_a$ be the subgraph of $G'$ containing those arcs corresponding to the arcs of $G_a$. Then $G_a$ and $G'_a$ are congruent, preserving the correspondence between their arcs.*

That is, we can rename the vertices and arcs of $G'_a$ so that arcs that formerly corresponded become identical. By the proof of case A,* $G_a$ and $G'_a$ are congruent, and the correspondence between their arcs is preserved if $\gamma'$ is of the form $\gamma'(b'c')$. Suppose not. Then $\gamma'$ is of the form $\gamma'(a'd')$. Let $\eta$ be an arc of $G$ with one end vertex in $G_a$. This vertex is either, $a$, $b$, $c$, or $d$, and $\eta$ touches $\alpha$ and $\beta$; $\alpha$, $\gamma$ and $\delta$; $\beta$ and $\gamma$; or $\delta$; but not other arcs of $G_a$. There is no way of adding an arc to $G'$ (without forming a 2-circuit) so that the similar conditions hold, and we have a contradiction, proving the lemma.

We return now to the theorem.

Case B. *Neither $G$ nor $G'$ is any of the graphs $G_a$, $G_b$, $G_c$.* In this case, we shall prove the stronger theorem, that $G$ and $G'$ are congruent, preserving the correspondence between their arcs. We shall consider first non-separable, then separable, graphs.

Case B1. *At least one of the graphs, say $G$, is non-separable.*

There are three cases to consider here.

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* We see, as in Case B1c, that $G'_a$ is connected.

Case B1a. *$G$ is of nullity* 0. By N, theorem 8, $G$ contains but a single arc, $ab$. Hence $G'$ contains the single arc $a'b'$, and $G$ and $G'$ are congruent.

Case B1b. *$G$ is of nullity* 1. Then, by N, theorem 10,* $G$ is a circuit, and contains therefore at least three arcs. Suppose first $G$ contains just three arcs. They are then of the form $\alpha(ab)$, $\beta(ac)$, $\gamma(bc)$. $\alpha'$ and $\beta'$ of $G'$ must touch, and they are of the form $\alpha'(a'b')$,$\beta'(a'c')$. $\gamma'$ must touch both $\alpha'$ $\beta'$, and is either of the form $\gamma'(b'c')$ or $\gamma'(a'd')$. But by the hypothesis of the theorem, $G'$ is not of the form $a'b'$, $a'c'$, $a'd'$; it is therefore of the form $a'b'$, $a'c'$, $b'c'$. $G$ and $G'$ are thus congruent, preserving the correspondence between their arcs.

Suppose next $G$ contains at least four arcs, $\alpha(ab)$, $\beta(bc)$, $\gamma(cd)$, $\delta(de)$, $\cdots$, $\zeta(fa)$. Let $\alpha'$, $\beta'$, $\gamma'$, $\delta'$, $\cdots$, $\zeta'$ be the corresponding arcs of $G'$. $\alpha'$ and $\beta'$ must touch, and they are therefor $\alpha'(a'b')$, $\beta'(b'c')$. $\gamma$ touches $\beta$ but not $\alpha$; the same must be true of $\gamma'$, and it is therefore $\gamma'(c'd')$. $\delta$ touches $\gamma$ but neither $\alpha$ nor $\beta$; hence $\delta'$ is $\delta'(d'e')$. We continue in this manner. Finally, $\zeta$ touches the preceding arc and $\alpha$, but none of the other arcs. The same must be true of $\zeta'$, and hence it is of the form $\zeta'(f'a')$. Thus $G$ and $G'$ are congruent as required.

Case B1c. *$G$ is of nullity $N > 1$.* We will assume the theorem is true if $G$ is of nullity $< N$, and will prove it for the case that $G$ is of nullity $N$, (including the case where $N = 2$). This will establish the proof for this case.

By N, theorem 18,† we can remove an arc or suspended chain $C$ from $G$, leaving a non-separable graph $G_1$ of nullity $N - 1$. Let $C'$ be the corresponding arc or arcs of $G'$. Remove them and any isolated vertices there may then be, forming the graph $G'_1$. As $G_1$ is connected, $G'_1$ is connected. For order the arcs of $G_1$ so that each arc other than the first touches one of the preceding arcs. Ordering the arcs of $G'_1$ in the same manner as the corresponding arcs of $G_1$, each arc other than the first touches one of the preceding arcs; thus $G'_1$ is connected. $C'$ is a chain. For if $C$ is a single arc, $C'$ is a single arc. Otherwise, the proof is exactly like the proof in the last part of case B1b.

We shall divide case B1c into two cases.

Case B1c$_1$. *None of the vertices of $C'$ other than its two end vertices are in $G'_1$.* We shall show first that $G_1$ and $G'_1$ are congruent, preserving the correspondence between their arcs. $G_1$ is not of the form $ab$, $ac$, $ad$, as it is non-separable. Suppose $G'_1$ were of this form. Then each pair of arcs

* P, theorem 6.

† P, theorem 7.

of $G_1$ have a common vertex, and $G_1$ is of the form $ab$, $ac$, $bc$. As $G$ contains no 2-circuit, $C$ contains at least two arcs. The first of these, $\delta$ say, has just one vertex in $G_1$, and touches therefor just two of the arcs of $G_1$. The corresponding arc $\delta'$ of $C'$ must touch just two of the arcs $a'b'$, $a'c'$, $a'd'$, and it is therefor of one of the forms $b'c'$, $b'd'$, $c'd'$. But by the hypothesis of this case, $\delta'$ has but a single end vertex in $G'$. This is a contradiction, so $G'_1$ is not of the form described.

Suppose neither $G_1$ nor $G'_1$ are one of the graphs $G_a$, $G_b$ or $G_c$ of case A. Then the hypotheses of case B1 are fulfilled, and $G_1$ is of nullity $N-1$, and hence $G_1$ and $G'_1$ are congruent as required.

Suppose next one of the graphs $G_1$, $G'_1$ was one of the graphs $G_a$, $G_b$, $G_c$. If $G_1$ is the graph $G_c$, $C$ contains an arc with just one end vertex in $G_c$, as $G$ contains no 2-circuit. But $G_c$ contains $G_a$ as a subgraph. Let $G'_a$ be the arcs of $G'$ corresponding to $G_a$. Then, by the lemma, $G_a$ and $G'_a$ are congruent, preserving the correspondence between their arcs. Hence, following the proof of case A, we see that $G_1$ and $G'_1$ are congruent as required. If, next, $G_1$ is the graph $G_b$, $C$ contains at least two arcs. For if $C$ contained but one arc, this arc would be $\zeta(ad)$, and $G$ would be the graph $G_c$. This is ruled out by the hypothesis of case B. Again, $G$ contains $G_a$ as a subgraph, and there is an arc with just one end vertex in $G_a$. Hence $G_a$ and $G'_a$, and therefor $G_1$ and $G'_1$, are congruent as required. Finally, $G_1$ is not $G_a$, for $G_a$ is separable. If now $G'_1$ were one of the graphs $G_a$, $G_b$, $G_c$, $G_1$ would be also, by case A, and we are back to the case we have considered. Thus in all cases $G_1$ and $G'_1$ are congruent as required.

Rename the vertices and arcs of $G'_1$ so that upon dropping primes, $G'_1$ and $G_1$ become identical. We can do this, preserving the correspondence between the arcs. Suppose the chain $C$ had $a$ and $b$ as end vertices in $G_1$. We shall show that the end vertices of $C'$ in $G'_1$ are $a'$ and $b'$. Let $A$ be the arcs of $G_1$ on $a$, and $B$, those on $b$. By N, theorem 8, there are at least two arcs in $A$, as $G_1$ is of nullity $N-1 \geqq 1$; thus a is the only vertex on them all. Similarly, $b$ is the only vertex on all the arcs $B$. Let $A'$, $B'$ be the corresponding arcs of $G'$. They are then on $a'$ and $b'$, respectively.

Suppose first $C$ contained but a single arc, $\alpha(ab)$. As it touches all the arcs of both $A$ and $B$, the corresponding arc $\alpha'$ of $C'$ touches all the arcs of both $A'$ and $B'$. Suppose $\alpha'$ were not $\alpha'(a'b')$. Say $\alpha'$ is not on $a'$. As it must touch all the arcs of $A'$, there are just two such arcs, $a'c'$ and $a'd'$, and $\alpha'$ is $\alpha'(c'd')$. Neither $c'$ nor $d'$ is $b'$. For then $G'_1$ would contain the arc $a'b'$, hence $G_1$ would contain the arc $ab$, and this arc with the arc $\alpha$ would form a 2-circuit. As $\alpha'(c'd')$ touches all the arcs of $B$, there are just

two such arcs, $b'c'$ and $b'd'$. There are no more arcs in $G'_1$. For if there were, as there are no more on $a'$ or $b'$, there must be arcs on $c'$ or $d'$. Hence there are arcs on $c$ or $d$ in $G_1$. But then $\alpha'$ would touch these arcs in $G'_1$, and $\alpha$ would not touch the corresponding arcs in $G_1$. Hence $G'$ is just the graph $G_b$. But this is ruled out by the hypotheses of case B. Therefore $\alpha'$ is $\alpha'(a'b')$, and $G$ and $G'$ are congruent as required.

Suppose next $C$ contains at least two arcs. The first arc, $\alpha$, has $a$ in common with the arcs $A$. As $C'$ has only its end vertices in $G'_1$, by hypothesis, the arc $\alpha'$ corresponding to $\alpha$ has just one vertex in $G_1$, and this vertex must be on all the arcs of $A'$. It is therefor the vertex $a'$. Similarly, the other end vertex of $C'$ is $b'$, and thus we can rename the vertices and arcs of $C'$ with primed letters so that, upon dropping primes, it becomes identical with $C$. Therefore $G$ and $G'$ are congruent, as required. This disposes of case $\text{B1c}_1$.

Case $\text{B1c}_2$. *$C'$ has a vertex besides its two end vertices in $G'_1$.* Then $C'$ contains just two arcs. For suppose $C'$ contained more than two arcs. Then $C$ would contain more than two arcs. None of these but the first and last touch arcs of $G_1$. Hence none of the arcs of $C'_1$ but the first and last have vertices in $G'_1$, and thus there would be no vertices of $C'$ but the first and last in $G'_1$.

Let $\alpha(ac)$, $\beta(cb)$ be the two arcs of $C$, and let $\alpha'(a'c')$, $\beta'(c'b')$ be those of $C'$. Then $a'$, $b'$ and $c'$ are all in $G'_1$. There is an arc $\gamma'(c'd')$ in $G'_1$, where $d'$ is neither $a'$ nor $b'$, as $G'$ contains no 2-circuit. This arc touches both $\alpha'$ and $\beta'$. Hence the corresponding arc $\gamma$ of $G$ touches both $\alpha$ and $\beta$, and is therefore $\gamma(ab)$. As $G_1$ is of nullity $N-1 \geqq 1$, $G$ contains more arcs. As $G_1$ is non-separable, there must be more arcs on both $a$ and $b$, for otherwise one of these vertices would be a cut vertex. Let $\delta(ad)$ be an arc on $a$. It touches $\alpha$ and $\gamma$ but not $\beta$. Hence the corresponding are $\delta'$ of $G'$ is $\delta'(a'd')$. Similarly, if $\epsilon(be)$ is an arc on $b$ in $G_1$, the corresponding arc in $G'_1$ is $\epsilon'(b'd')$. As now $\delta'$ and $\epsilon'$ touch, $\delta$ and $\epsilon$ touch, and thus $e$ is $d$. $G$ contains no more arcs. For if it did, it would contain an arc on one of the vertices $a$, $b$, $d$. But we cannot fit a similar arc into $G'$. $G$ is therefore the graph $G_b$. But this was ruled out, so case $\text{B1c}_2$ does not exist. Case B1c, and hence case B1, is now disposed of.

Case B2. *Both $G$ and $G'$ are separable.* Let $H_1, \cdots, H_m$ be the components of $G$, and let $H'_1, \cdots, H'_n$ be those of $G'$. Let $I'_i$ be the subgraph of $G'$ containing those arcs corresponding to the arcs of $H_i$, $i = 1, \cdots, m$, and let $I_j$ be the subgraph of $G$ containing those arcs corresponding to the arcs of $H'_j$, $j = 1, \cdots, n$.

Case B2a. One of the graphs $I_1, \cdots, I_n, I'_1, \cdots, I'_m$, say $I'_1$, is of the form $\alpha'(a'b')$, $\beta'(a'c')$, $\gamma'(a'd')$. Then $H_1$, being non-separable, is of the form $\alpha(ab)$, $\beta(ac)$, $\gamma(bc)$. There are other arcs in $G$, and hence there is an arc on one of the vertices $a, b, c$. Let such an arc be $\delta(ad)$. The corresponding arc $\delta'$ in $G'$ is $\delta'(b'c')$. There are no more arcs in $G'$. For suppose there were an arc with both end vertices in $I'_1$. It is then either $\epsilon'(b'd')$ or $\zeta'(c'd')$, say the first. The corresponding arc in $G$ must be $\epsilon(bd)$. But then $H_1$ is not a component of $G$, for $\alpha(ab)$, $\epsilon(bd)$, $\delta(ad)$ is a circuit with arcs in $H_1$ and arcs not in $H_1$, contrary to N, theorem 17.* There cannot be an arc with just one end vertex in $I'_1$, for the lemma would be contradicted. Thus case B2a does not exist.

Case B2b. *No one of the graphs* $I_1, \cdots, I_n, I'_1, \cdots, I'_m$ *is of the form* $ab, ac, ad$. Consider first any one of the graphs $H_1, \cdots, H_m$, say $H_1$, which is one of the graphs $G_b$, $G_c$. (No one is $G_a$, as $G_a$ is separable.) Consider that part of $H_1$, $G_a$. As $G$ is connected, there is an arc with just one end vertex in $G_a$ (i. e. in $H_1$). Hence, by the lemma, $G_a$ and the corresponding arcs of $I'_1$, $G'_a$, are congruent, preserving the correspondence between their arcs. Hence, as is seen from the proof of case A, $H_1$ and $I'_1$ are congruent similarly. Consider next any component, say $H_2$, which is not either $G_b$ or $G_c$. Then the corresponding graph, $I'_2$, is neither of these graphs, for otherwise, case A would be contradicted. As $H_2$ is connected, $I'_2$ is connected. Thus, by case B1, $H_2$ and $I'_2$ are congruent, preserving the correspondence between their arcs. Thus $H_i$ and $I'_i$ are congruent as required, $i = 1, \cdots, m$. Similarly, $H'_j$ and $I_j$ are congruent as required, $j = 1, \cdots, n$.

Now $I'_1$ is a non-separable subgraph of $G'$, and is therefor contained in one of the components $H'_1, \cdots, H'_n$, say $H'_1$, of $G'$, by N, theorem 11. Similarly, $I_1$ is contained in one of the components $H_1, \cdots, H_m$, say $H_j$, of $G$. Hence $H_1$ is contained in $H_j$, and thus they are identical. Thus the arcs of $H_1$ and $H'_1$ correspond, and they are congruent, preserving this correspondence. Similarly for the other components. Rename the vertices and arcs in each $H'_i$ with the subscript $i$ and primes so $H'_i$ becomes identical with $H_i$ when these subscripts and primes are dropped.

Let $H_1$ and $H_2$ be two components of $G$ with a common vertex, $a$, (they have but one common vertex, as otherwise they would form a circuit of graphs, contrary to N, theorem 17 *), and let $H_{12}$ be the graph containing the arcs of both. Let $H'_1$, $H'_2$, $H'_{12}$ be the corresponding subgraphs of $G'$. We shall show that $H_{12}$ and $H'_{12}$ are congruent as required, supposing only that

* P, theorem 5.

$H_1$, $H_2$, $H'_1$, $H'_2$ are connected. We next show that $H_{123}$, composed of $H_{12}$ and $H_3$ together, is congruent as required with the corresponding subgraph $H'_{123}$ of $G'$. Continuing, we find finally that $G$ and $G'$ are congruent as required.

Let $A_1$ and $A_2$ be the arcs of $H_1$ and $H_2$ on $a$. Let $A'_1$ and $A'_2$ be the corresponding arcs of $H'_1$ and $H'_2$. Then the arcs $A'_1$ are all on $a'_1$, and the arcs $A'_2$ are all on $a'_2$. We must show that $a'_1$ and $a'_2$ are the same vertex in $G'$. Each arc in $A_2$ touches an arc in $A_1$, and hence each arc in $A'_2$ touches an arc in $A'_1$. Therefor $H'_1$ and $H'_2$ have a common vertex. Call it $x'$. Moreover, any arc in $A'_2$ has just one vertex in $H'_1$, as $H'_1$ and $H'_2$ have at most a single vertex in common, being two components of $G'$.

Suppose first there is but a single arc in $H'_1$, $a'_1b'_1$. Then if $a'_1$ is not $x'$, $b'_1$ is $x'$, and we interchange $a'_1$ and $b'_1$ so that $a'_1$ becomes $x'$. Suppose next $H'_1$ contains at least two arcs. If there are two arcs in $A'_1$, they have only the vertex $a'_1$ in common, and hence any arc in $A'_2$ has this vertex as an end vertex; that is, $x'$ is $a'_1$. If $A'_1$ contains but a single arc $a'_1b'_1$, there are arcs on $b'_1$, as $H'_1$ is connected. $ab$ being the corresponding arc of $H_1$, there are corresponding arcs on $b$. The arcs of $A_2$ touch $ab$ but not these other arcs on $b$. Hence the arcs of $A'_2$ touch $a'_1b'_1$ but not these other arcs on $b'$. They are therefore on the vertex $a'_1$; that is, $x'$ is $a'_1$.

Thus in all cases, if $x'$ is not $a'_1$, we can make it so, preserving the correspondence between the arcs of $G$ and $G'$. Similarly, if $x'$ is not $a'_2$, we can make it so. Hence, if $a'_1$ is not $a'_2$, we can make it so. As finally, $H'_1$ and $H'_2$ have but one common vertex, $H_{12}$ and $H'_{12}$ are congruent as required.

The only fact we need notice to complete the proof is that if $I_1$ and $I_2$ are two connected groups of components of $G$, they have at most a single vertex in common, and the same is true for $G'$. For otherwise, we could find a circuit entering both of these groups of components, contrary to N, theorem 17.* The proof for case B2b, therefore for case B2, therefore for case B, and therefore of the theorem, is now complete.

THEOREM 2. *Let $G$ and $G'$ be two triply connected graphs, and let there be a* 1-1 *correspondence between their arcs so that to any set of arcs forming a circuit in one graph, corresponds a set of arcs forming a circuit in the other. Then $G$ and $G'$ are congruent.*

For a definition of "triply connected," see the next section. We shall show that to any two arcs that touch in one graph, there correspond two arcs that touch in the other. Theorem 1 then applies.

* P, theorem 5.

Let $\alpha(ab)$, $\beta(bc)$ be any two arcs in $G$ with the common vertex $b$. We shall show that the corresponding arcs $\alpha'$, $\beta'$ in $G'$ have a common vertex. As $G$ is triply connected, we cannot disconnect it without dropping out at least three vertices. Therefore if we drop out the vertex $b$ and the arcs on it, forming the graph $G_1$, we cannot disconnect this graph without dropping out at least two vertices; that is, $G_1$ has no cut vertex. Therefore, by N, theorems 5 and 7,* the vertices $a$ and $c$ are contained in a circuit $P$ in $G_1$. $P$ consists of two chains $B$ and $C$, joining $a$ and $c$. Now put back $b$ and the arcs on $b$. The two arcs $ab$, $bc$ form a third chain, $A$, joining $a$ and $c$. The two chains $A$ and $C$ form a circuit $Q$, and the two chains $A$ and $B$ form a circuit $R$. Moreover, no other subset of the arcs of these three chains form a circuit.

The arcs in $G'$ corresponding to these three circuits form three circuits, $P'$, $Q'$, $R'$. Suppose now the arcs $\alpha'$ and $\beta'$ had no common vertex. These arcs we shall call $A'$. Let $B'$ be those arcs corresponding to $B$, and $C'$, those corresponding to $C$. Let $\alpha'$, $\beta'$ be the arcs $\alpha'(a'b')$, $\beta'(c'd')$.

Now as $A'$ and $B'$ together form a circuit, $B'$ consists of two chains, one of them joining $a'$ and either $c'$ or $d'$, say $d'$, and the other joining $b'$ and $c'$. Similarly, $C'$ is in two chains; either one joins $a'$ and $d'$ and the other joins $b'$ and $c'$, or one joins $a'$ and $c'$ and the other joins $b'$ and $d'$. The first case cannot be, as then there would be a chain in both $B'$ and $C'$ joining $a'$ and $d'$, and these two chains, or parts of them, form a circuit; but the whole of $B'$ and $C'$ forms a circuit, and therefore no part of them does. Therefore the second case occurs. Consider the following three chains: One part of $A'$, $\alpha'(a'b')$, one part of $B'$, that part joining $b'$ and $c'$, and one part of $C'$, that part joining $c'$ and $a'$. As $B'$ and $C'$ together form a circuit, they have no vertices in common other than the end vertices of the chains forming them. Hence the three chains described form a circuit, containing arcs of $A'$, $B'$ and $C'$. The corresponding arcs of $G$ must form a circuit. But this is not the case, as we have seen, and we have a contradiction. Hence $\alpha'$ and $\beta'$ have a vertex in common, and the theorem is proved.

This theorem is not true for all non-separable graphs. An example is given by the two graphs $G'$, $G''$, given in N, after the proof of theorem 22.

THEOREM 3. *Let there be a* 1-1 *correspondence between the arcs of the triply connected graphs $G$ and $G'$ so that to any set of arcs of one graph forming a subgraph of nullity 1 there corresponds a set of arcs of the other graph forming a subgraph of nullity* 1. *Then $G$ and $G'$ are congruent.*

We shall prove that circuits correspond to circuits. Theorem 2 then

* P, theorem 1.

applies. Consider any circuit $P$ of $G$. It is of nullity 1. Hence the corresponding arcs $P'$ of $G'$ are of nullity 1. Now drop out any arc from $P$. The nullity is reduced to 0. Hence the corresponding arcs of $G'$ cannot be of nullity 1, and are therefore of nullity 0. Therefore, by N, theorem 9, $P'$ is a circuit. Similarly, if $P'$ is a circuit in $G'$, the corresponding arcs $P$ form a circuit in $G$.

2. *The connectivity of graphs.* In this section, we allow the graphs to contain 2-circuits, but no 1-circuits.

*Definitions.* Let $G$ be a graph containing at least $n+1$ vertices, such that it is impossible to drop out $n-1$ or fewer vertices and the arcs on them in such a manner that the resulting graph is not connected. We shall say then that $G$ is $n$-tuply connected. (We consider only numbers $n \geqq 1$). If $G$ is $n$-tuply connected but not $(n+1)$-tuply connected, we say its connectivity is $n$.

For $n=1$, we have: A graph $G$ is simply connected if it contains at least two vertices, and it cannot be disconnected by dropping out no vertices; i. e., $G$ itself is connected. For $n=2$, N, theorems 5 and 6 give: a non-separable graph containing at least three vertices is doubly connected, and conversely.

THEOREM 4. *A necessary and sufficient condition that a graph $G$ be $n$-tuply connected is that $G$ contain at least $n+1$ vertices, and there exist no two graphs $H'$ and $H''$ such that for some number $k \leqq n-1$, $H'$ contains the vertices $a'_1, \cdots, a'_k$, $b'$, $H''$ contains the vertices $a''_1, \cdots, a''_k$, $b''$, and if we let $a'_i$ and $a''_i$ coalesce, forming the vertex $a_i$, $i=1, \cdots, k$, we thereby form $G$.*

For $n=1$, the theorem is: A necessary and sufficient condition that $G$ be simply connected is that $G$ contain at least two vertices, and is not formed of two connected parts $H'$, $H''$; i. e. $G$ is connected. (For we let $k=n-1=0$ vertices of $H'$ and $H''$ coalesce to form $G$.) The theorem is trivial for this case.

Consider now numbers $n>1$. The theorem is trivial if $G$ contains fewer than $n+1$ vertices; assume therefore $G$ contains at least $n+1$ vertices. Suppose first $G$ is formed from two graphs, $H'$, $H''$ as described. Drop out the vertices $a_1, \cdots, a_k$, and the arcs on them. As $H'$ and $H''$ had only these vertices in common, $b'$ and $b''$ are now in different connected parts. Hence $G$ is not $n$-tuply connected.

Suppose now $G$ is not $n$-tuply connected. Then we can drop out $k \leqq n-1$ vertices $a_1, \cdots, a_k$, so that there remain two vertices $b'$, $b''$, which

are in different connected parts in the resulting graph $G_1$. Let $H'_1$ be that part of $G_1$ containing $b'$, and let $H''_1$ be the rest of $G_1$. Replace each vertex $a_i$ of $G$ by the two vertices $a'_i$, $a''_i$ $(i = 1, \cdots, k)$ and replace the arcs we removed in the following manner: An arc $a_ic$, where $c$ is in $H'_1$, we replace by the arc $a'_ic$. An arc $a_id$, where $d$ is in $H''_1$, we replace by the arc $a''_id$. An arc $a_ia_j$ we replace by the arc $a'_ia'_j$. In this manner we have added the vertices $a'_1, \cdots, a'_k$ to $H'_1$, forming the graph $H'$, and we have added the vertices $a''_1, \cdots, a''_k$ to $H''_1$, forming the graph $H''$. Now $H'$ contains the vertex $b'$ distinct from $a'_1, \cdots, a'_k$, $H''$ contains the vertex $b''$ distinct from $a''_1, \cdots, a''_k$, and letting $a'_i$ and $a''_i$ coalesce $(i = 1, \cdots, k)$ forms $G$. This completes the proof.

THEOREM 5. *If the graph $G$ containing at least two vertices can be disconnected by dropping out $n-1$ or fewer arcs, it is not $n$-tuply connected.*

The theorem is trivial if $n = 1$. Assume $n > 1$. Drop out only just enough arcs to disconnect $G$. The resulting graph is then in two parts, $H$ and $H'$, and each arc we dropped out joins a vertex in $H$ to a vertex in $H'$. Let these arcs be $a_1b_1$, and $a_2b_2, \cdots, a_k\, b_k$, where $a_1, \cdots, a_k$ are in $H$ and $b_1, \cdots, b_k$ are in $H'$. Then $k < n$. The vertices $a_1, \cdots, a_k$, and also the vertices $b_1, \cdots, b_k$, may not all be distinct.

Case 1. There are two vertices $a_i$ and $b_j$ which are joined by no arc. Consider first the arc $a_1b_1$. As it does not join $a_i$ and $b_j$, either $a_1$ is distinct from $a_i$ or $b_1$ is distinct from $b_j$, say the first. Drop out $a_1$ and the arcs on it. Consider next the arc $a_2b_2$. One of the vertices $a_2$, $b_2$, is distinct from both $a_i$ and $b_j$. Drop out this vertex, if it is not already dropped out, and its arcs. Continue in this manner. At the end of the process, we have dropped out fewer than $n$ vertices, and we have dropped out all the arcs $a_1b_1, \cdots, a_k\, b_k$. The vertices $a_i$ and $b_j$ are still in the graph, but are joined by no chain. We have thus dropped out less than $n$ vertices and their arcs, disconnecting $G$, and hence $G$ is not $n$-tuply connected.

Case 2. Each vertex $a_i$ is joined to each vertex $b_j$ by an arc. Say there are $p$ distinct vertices in the set $a_1, \cdots, a_k$, and $q$ distinct vertices in the set $b_1, \cdots, b_k$. There are then $pq$ arcs in the set $a_1b_1, \cdots, a_kb_k$, or more, if $G$ contains 2-circuits. Now as $k < n$,

$$pq < n.$$

As

$$pq + 1 - (p + q) = (p - 1)(q - 1) \geqq 0, \; p + q \leqq pq + 1 < n + 1.$$

If $G$ contains $n$ or fewer vertices, it is not $n$-tuply connected. Assume it

contains at least $n+1$ vertices. Then, as there are but $p+q$ vertices in the set $a_1, \cdots, a_k, b_1, \cdots, b_k$, there is a vertex $c$ in $G$ not in this set.

Dropping out the arcs $a_1b_1, \cdots, a_kb_k$ from $G$ leaves the two connected parts $H$ and $H'$, one of which, say $H$, contains the vertex $c$. Drop out then the vertices $a_1, \cdots, a_k$ and their arcs from $G$. We have dropped out fewer than $n$ vertices and their arcs, disconnecting $G$, as $b_1$ and $c$ are joined by no chain, and thus $G$ is not $n$-tuply connected.

THEOREM 6. *If a vertex and its arcs be dropped out of the $n$-tuply connected graph $G$, the resulting graph $G'$ is $(n-1)$-tuply connected.*

For $G'$ contains at least $n$ vertices, and we must drop out at least $n-1$ vertices to disconnect it.

THEOREM 7. *A necessary and sufficient condition that a graph containing no 2-circuit be $n$-tuply connected is that any two of its vertices be joined by $n$ distinct chains.**

For $n=1$, the theorem is obvious. Suppose $n=2$. If the graph contains but two vertices, the theorem is obvious. If the graph contains at least three vertices, the theorem becomes theorem 7 of N.† Assume the theorem is true for all numbers $1, 2, \cdots, n-1$; we shall prove it for $n=n$. This will establish it in general.

We shall prove first the sufficiency of the condition. To show that $G$ contains at least $n+1$ vertices, let $a$ and $b$ be any two of its vertices; they are joined by $n$ distinct chains. As there is at most a single arc $ab$, there must be $n-1$ distinct chains joining $a$ and $b$, each of which contains at least two arcs. Each chain contains at least one vertex in its interior, so there are at least $n-1+2=n+1$ vertices in $G$. Now suppose we could disconnect $G$ by dropping out $k<n$ vertices. Some two remaining vertices, $a$, $b$, are now joined by no chain. Hence in the original graph each chain joining them must pass through one of the $k$ vertices we dropped out, and there are not $n$ distinct chains therefore, a contradiction.

To prove the necessity of the condition, we must show that for $n>2$, if $G$ is an $n$-tuply connected graph and $a$, $f$ are any two of its vertices, then $a$ and $f$ are joined by $n$ distinct chains. As $n>2$, $G$ is non-separable, and

* i. e. chains which have only the two given vertices in common. Similar theorems have been proved by K. Menger, "Zur allgemeinen Kurventheorie," *Fundamenta Mathematicae*, Vol. 10 (1926), pp. 96-115, Satz $\beta$; N. E. Rutt, "Concerning the Cut-Points of a Continuous Curve When the Arc Curve $ab$ Contains Exactly $n$ Independent Arcs," *American Journal of Mathematics*, Vol. 51 (1929), pp. 217-246.

† P, theorem 1.

hence, by N, theorem 7,* there is a circuit passing through $a$ and $f$. One of the two chains joining them, say the chain $D = ab, bc, cd, \cdots, ef$, contains at least two arcs. Form the graph $G_2$ from $G$ by dropping out the vertex $b$ and the arcs on it. By theorem 6, $G_2$ is $(n-1)$-tuply connected, and hence there are $n-1$ distinct chains joining $a$ and $c$. Now replace the vertex $b$ and the arcs on $b$. The arcs $ab$, $bc$ form another chain from $a$ to $c$, so that there are in $G$ $n$ distinct chains joining $a$ and $c$. Now if $f$ is not joined to $a$ by $n$ distinct chains, there is a first vertex of the chain $D$ (coming after $c$), say $d$, which is not, while the vertex preceding it, say $c$, is. We shall show this leads to a contradiction.

Let $A_1, A_2, \cdots, A_n$ be $n$ distinct chains from $a$ to $c$. Form the graph $G_1$ from $G$ by dropping out the vertex $c$ and the arcs on $c$. There are in $G_1$ $n-1$ distinct chains $B_1, B_2, \cdots, B_{n-1}$ from $d$ to $a$. We shall find in $G$ $n$ distinct chains $C_1, C_2, \cdots, C_n$ from $d$ to $a$.

Case 1. $d$ is not on any of the chains $A_1, \cdots, A_n$. The method is to find first a set of chains $C'_1, \cdots, C'_{n-1}$, with the following properties:

(1) For some number $l$, $0 \leqq l \leqq n-1$, $l$ of the chains, say $C'_{n-l}$, $C'_{n-l+1}, \cdots, C'_{n-1}$ are identical with the chains $B_{n-l}, B_{n-l+1}, \cdots, B_{n-1}$.

(2) Each other chain $C'_i$, (of which there are $n-1-l$), is identical with a part of $B_i$, that part stretching from $d$ to a certain vertex, say $b_i$, of one of the chains $A_1, \cdots, A_n$, say $A_i$, and these vertices lie on distinct chains.

(3) No one of the chains $C'_i$ has a vertex in common with either a vertex of that part of any chain $A_j$ lying between $a$ and $b_j$, $j = 1, 2, \cdots, n-1-l$, or with any vertex of any of the chains $A_{n-l}, A_{n-l+1}, \cdots, A_n$, other than the vertex $a$. (Of course no chain $B_i$ passes through the vertex $c$).

We can then construct the chains $C_1, \cdots, C_n$ as follows: For $i = 1, 2, \cdots, n-1-l$, $C_i$ consists of $C'_i$ plus that part of $A_i$ lying between $a$ and $b_i$; for $j = n-l, n-l+1, \cdots, n-1$, $C_j$ is exactly $C'_j$; and $C_n$ consists of the arc $cd$ plus the chain $A_n$. These are $n$ distinct chains from $d$ to $a$, as required.

We shall find the chains $C'_1, \cdots, C'_{n-1}$ in the following manner: First follow $B_1$ from $d$ towards $a$ till we reach a vertex of one of the chains $A_1, \cdots, A_n$, or the vertex $a$ itself. This much of $B_1$ might serve as $C'_1$. Next follow $B_2$ towards $a$, and alter $C'_1$ if necessary, so we have two chains that might serve as $C'_1$, $C'_2$. Next follow $B_3$ towards $a$, and alter the other two chains if necessary till we have three distinct chains which might serve as $C'_1$, $C'_2$, $C'_3$. Continue in this manner till we have found the $n-1$ re-

* P, theorem 1.

quired chains. At each stage therefore, the properties (1), (2), (3) hold, with $n-1$ replaced by some number $k \leqq n-1$.

To turn to the actual construction, we find $C'_1$ as described. Next follow $B_2$ from $d$ towards $a$. Suppose first $B_1$ was entirely distinct from the chains $A_1, \cdots, A_n$, so that $C'_1$ is $B_1$. If we reach $a$ before touching any of the chains $A_1, \cdots, A_n$, $B_2$ forms $C'_2$. If we reach first a vertex $b_2$ of the chain $A_2$, say, this much of $B_2$ forms $C'_2$. In either case, (1), (2) and (3) hold. Suppose next $B_1$ had the vertex $b_1$ in common with $A_1$ say. If we reach either the vertex $a$ or a vertex $b_2$ of one of the chains $A_2, \cdots, A_n$, say $A_2$, before reaching any other vertex of the chains $A_2, \cdots, A_n$, or any vertex of $A_1(ab_1)$, (that is, that part of $A_1$ lying between $a$ and $b_1$), this much of $B_2$, that is, $B_2$ itself or $B_2(db_2)$, forms $C'_2$, and again (1), (2) and (3) hold. We are thus in difficulty only if we reach a vertex $b'_2$ of $A_1(ab_1)$ before reaching either $a$ or a vertex of one of the chains $A_2, \cdots, A_n$.

In this case, let $C'_2$ be $B_2(db'_2)$. Let us follow $B_1$ further from $b_1$ towards $a$. If we reach either $a$ or a vertex $b'_1$ of one of the chains $A_2, \cdots, A_n$ before reaching a vertex of $A_1(ab'_2)$, this much of $B_1$ added to the original $C'_1$ forms the new $C'_1$, and again (1), (2) and (3) hold. We are in difficulty only if we reach first a vertex, say $b''_1$, of $A_1(ab'_2)$. In this case, let $B_1(db''_1)$ be the new $C'_1$. Follow now $B_2$ from $b'_2$ towards $a$, forming more of the new $C'_2$. Again, we are in difficulty only if we reach a vertex of $A_1(ab''_1)$ before reaching either $a$ or a vertex of one of the chains $A_2, \cdots, A_n$. Continuing, the process must at some time come to an end, as there are only a finite number of vertices in the chain $A_1$. We now have two chains $C'_1$, $C'_2$, as required.

We shall assume now we have found $k-1$ chains $C'_1, \cdots, C'_{k-1}$, as required, and we shall find the $k$'th chain $C'_k$. We can rename chains and vertices so that for some number $l \leqq k-1$, $l$ of the chains, $C'_{k-l}$, $C'_{k-l+1}$, $\cdots$, $C'_{k-1}$ are identical with $B'_{k-l}$, $B'_{k-l+1}, \cdots, B'_{k-1}$, respectively, and each other chain $C'_i$ is identical with $B_i(db_i)$, $i = 1, 2, \cdots, k-1-l$, where $b_i$ lies on the chain $A_i$.

Follow $B_k$ from $d$ towards $a$. If we reach either the vertex $a$ or a vertex $b_{k-l}$ of one of the chains $A_{k-l}, A_{k-l+1}, \cdots, A_n$, say $A_{k-l}$, before reaching any other vertex of these chains or any vertex of $A_i(ab_i)$, $i = 1, 2, \cdots, k-1-l$, this much of $B_k$ forms $C'_k$, and (1), (2) and (3) hold. We are in difficulty only if we reach first a vertex $b'$ of $A_i(ab_i)$, $1 \leqq i \leqq k-1-l$. Let this much of $B_k$ be $C'_k$. We alter $C'_i$ as follows: Follow $B_i$ from $b_i$ towards $a$. If we reach $a$ or a vertex of one of the chains $A_{k-l}, \cdots, A_n$ first, this much of $B_i$ together with $B_i(db_i)$ forms the new $C'_i$, and again (1), (2) and (3) hold. Suppose we reached first the vertex $b''$ of $A_j(ab_j)$, where $1 \leqq j \leqq$

$k-1-l$, $j \neq i$, or of $A_i(ab')$. We then follow $B_j$ or $B_k$, as the case may be, further towards $a$, forming more of the new $C'_j$ or $C'_k$. Again, we are in difficulty only if we reach a vertex of $A_s$, $1 \leqq s \leqq k-1-l$, lying between $a$ and the vertex $b_s$ (or $b'$ or $b''$). In this case, we follow $B_s$ (or $B_k$ or $B_i$) further towards $a$. The process must eventually come to an end, as there are but a finite number of vertices in the graph. We now have the required chains $C'_1, \cdots, C'_k$.

Putting $k = n-1$, we have the required $n-1$ chains $C'_1, \cdots, C'_{n-1}$, and the proof for case 1 is complete.

Case 2. $d$ is on one of the chains $A_1, \cdots, A_n$, say $A_n$. Let $A_n$ now stand for only that part of the former $A_n$ lying between $a$ and $d$. Exactly as we before found $n-1$ distinct chains from $d$ to $a$ which did not touch one of the chains $A_1, \cdots, A_n$ (which chain we called $A_n$), we now find a similar set of chains. If, first, $A_n$ is a chain we have not touched, it forms the required $n$-th chain from $d$ to $a$. If $A_i$, $i \neq n$, was a chain we have not touched, this chain plus the arc $cd$ forms the required $n$-th chain. The proof for case 2 is now complete also.

3. *Applications to dual graphs.* In this section we remove the restriction on the graphs we consider, that they contain no 1- or 2-circuits, for during the proofs of the following theorems we may run into graphs with such circuits, and we do not wish to have to avoid them.

A planar graph cannot be more than quintuply connected. For suppose there were a sextuply connected graph. Then there is such a graph containing no 2-circuits. By theorem 5, each vertex is on at least six arcs. But this cannot be, as is easily seen from Euler's formula.* However, there are quintuply connected planar graphs, for instance, the dual graph of the dodecahedron (containing twelve vertices, each of which is on five arcs).

Dual graphs may not have the same connectivity. (The dual of the graph just mentioned is but triply connected, each vertex being on but three arcs.) However, we might define a certain notion, the connectivity of a graph *in the large*. Then to a certain extent, we have the theorem that *dual graphs have the same connectivity in the large*. This will be understood on referring to theorem 9 below. We will not go further into the matter here.

THEOREM 8. *Let $G$ and $G'$ be connected† dual graphs. Let $a$ be a*

* This is the equivalent of the theorem that a map on a sphere contains at least one region with less than six sides.

† The hypothesis that the graphs be connected is obviously unnecessary. We include it merely for convenience.

*vertex of $G$, and let $A$ be the arcs on $a$. Let $A'$, the corresponding arcs of $G'$, form a circuit. Let $b'$ and $c'$ be two distinct vertices of this circuit, dividing it into the two chains $B'$ and $C'$. Let $B$ and $C$ be the corresponding arcs of $G$. Form $G'_1$ from $G'$ by letting $b'$ and $c'$ coalesce into the vertex $a'$. Form $G_1$ from $G$ by replacing $a$ by the two vertices $b$ and $c$, and letting the arcs $B$ end on $b$, and the arcs $C$, on $c$. Then $G_1$ and $G'_1$ are duals, preserving the correspondence between their arcs.*

Let $R$, $R'$, $r$, $r'$, etc., stand for the ranks of $G$, $G'$, $H$, $H'$, etc., respectively. Letting $b'$ and $c'$ coalesce in $G'$ reduced the number of vertices by one, and hence

$$R'_1 = R' - 1.$$

Let $H_1$ be any subgraph of $G_1$, and let $H'_1$ be the complement of the corresponding subgraph of $G'_1$. We must show that

$$r'_1 = R'_1 - n_1.$$

Let $H$ and $H'$ be the subgraphs of $G$ and $G'$ containing the same arcs as $H_1$ and $H'_1$, respectively. Then, as $G$ and $G'$ are duals,

$$r' = R' - n.$$

Case 1. $b'$ and $c'$ are in different connected pieces in $H'$. Then $b$ and $c$ are in the same connected piece in $H_1$. For add to $H'$ every arc of $G'$ we can without connecting $b'$ and $c'$, forming the graph $I'$. Let $D'$ be the remaining arcs of $G'$, and let $D$ be the corresponding arcs in $G$. Then $D$ is the complement in $G$ of the arcs corresponding to $I'$ in $G'$. $I'$ is in two connected pieces, while adding any arc of $D'$ renders the graph connected. Therefore, as $G'$ is a dual of $G$, the arcs $D$ are of nullity 1, while dropping out any one of them reduces the nullity to 0. Hence, by N, theorem 9, $D$ is a circuit. As $b'$ and $c'$ are not connected in $I'$, at least one of the arcs of $B'$ and at least one of the arcs of $C'$ are missing from $I'$, and therefore at least one of the arcs of $B$ and at least one of the arcs of $C$ are present in $D$. In forming $G_1$ from $G$, the circuit $D$ is broken at the vertex $a$, and becomes a chain $D_1$ in $G_1$. As $D_1$ contains still an arc of $B$ and an arc of $C$, $D_1$ joins $b$ and $c$. As $I'$ contains every arc of $H'$, $D_1$ is contained in $H_1$. Hence $b$ and $c$ are connected in $H_1$.

In forming $H'_1$ from $H'$ by letting $b'$ and $c'$ coalesce, the numbers of vertices and of connected pieces are each reduced by 1, and hence

$$r'_1 = r'.$$

In forming $H$ from $H_1$ by letting $b$ and $c$ coalesce, the the number of vertices

is reduced by 1, while the number of connected pieces and of arcs remains the same. Hence

$$n_1 = n - 1.$$

These equations with the equations above give

$$r'_1 = R'_1 - n_1,$$

as required.

Case 2. $b'$ and $c'$ are connected in $H'$. Then $b$ and $c$ are not connected in $H_1$. For suppose there were a chain $D_1$ joining them. The arcs of this chain form a circuit $D$ in $G$, containing one arc of $B$ and one arc of $C$. Let $D'$ be the corresponding arcs of $G'$, and let $I'$ be the complement of $D'$. Then as $G'$ is a dual of $G$, $I'$ is in two connected pieces, while adding any arc of $D'$ connects the graph. Let $d'e'$ be the arc of $B'$ contained in $D'$, where $d'$ is nearest $b'$, $e'$ is nearest $c'$. ($d'$ may be $b'$, and $e'$ may be $c'$). As $D'$ contains no other arc of $B'$, $b'$ and $d'$ are connected in $I'$, as are $c'$ and $e'$. Adding the arc $d'e'$ to $I'$ connects $I'$. Therefore $d'$ and $e'$, and therefore also $b'$ and $c'$, were formerly not connected. As $I'$ contains every arc of $H'$, $b'$ and $c'$ are not connected in $H'$ either, a contradiction.

As $b'$ and $c'$ are connected in $H'$, letting them coalesce reduces the rank of $H'$, that is,

$$r'_1 = r' - 1.$$

As $b$ and $c$ are not connected in $H'$, letting them coalesce does not alter the rank or nullity of $H_1$, that is,

$$n_1 = n.$$

These equations with the first equations give

$$r'_1 = R'_1 - n_1$$

again, as required.

THEOREM 9. *Let $G$ be a non-separable planar graph with the following properties. If the vertices $a_1, a_2, \cdots, a_n$ and their arcs are dropped out, the resulting graph is in two connected pieces, $H_1$ and $H_2$, while no proper subset of these vertices, if dropped out, disconnect the graph. Either $H_1$ is of nullity $> 0$ or one of the vertices $a_i$ is joined to $H_1$ by at least two arcs, and the same is true of $H_2$. Then if $G'$ is a dual of $G$, $G'$ can be disconnected by dropping out a corresponding set of $n$ vertices and their arcs.*

A definition of a "corresponding set of vertices" is given during the proof.

Let $A_i$ be the arcs on $a_i$, let $A_i(H_1)$ be those arcs joining $a_i$ to vertices in $H_1$, let $A_i(H_2)$ be those joining $a_i$ to vertices in $H_2$, and let $A_i(A)$ be

those joining $a_i$ to other vertices of the set $a_1, \cdots, a_n$, $i = 1, \cdots, n$. There are arcs in $A_i(H_1)$ and $A_i(H_2)$ for each $i$. For suppose for instance there were no arcs in $A_1(H_1)$. Then if we drop out the vertices $a_2, \cdots, a_n$ and their arcs from $G$, $a_1$ is now connected only to $H_2$, and hence $H_1$ and $H_2$ are not connected, contrary to hypothesis. Let $A'_i$, $A'_i(H_1)$, $A'_i(H_2)$, $A'_i(A)$ be the arcs of $G'$ corresponding to $A_i$, $A_i(H_1)$, $A_i(H_2)$, $A_i(A)$ respectively, $i = 1, \cdots, n$.

The arcs $A_i$, if dropped out, disconnect $G$.† Hence the arcs $A'_i$ form a circuit, $i = 1, \cdots, n$. Let us form $G^*_1$ from $G$ by dropping out the arcs $P = A_1(H_1), A_2(H_1), \cdots, A_n(H_1)$. $G^*_1$ is in two connected pieces, one of them being $H_1$, the other being $H_2$ together with the vertices $a_1, \cdots, a_n$. Putting back any of these arcs connects the two pieces. Hence the corresponding arcs of $G'$ form a circuit $P'$.

The arcs $A'_i(H_1)$ for any $i$ form a chain. For suppose not. Then for some $i$, $A'_i(H_1)$ is composed of $k$ chains $C'_1, \cdots, C'_k$. The rest of the circuit $A'_i$ is composed of $k$ chains $D'_1, \cdots, D'_k$, and each chain $D'_j$ joins some two chains of the set $C'_1, \cdots, C'_k$. We formed $G^*_1$ from $G$ by dropping out the arcs $P$. Let us form $G^*_2$ by dropping out also the arcs corresponding to the arcs of $D'_1$. As $D'_1$ has its end vertices in $P'$, $P'$ and $D'_1$ are together of nullity 2. Hence $G^*_2$ must be in three connected pieces. But this is not so. For the arcs of $A_i(H_2)$ and $A_i(A)$ all join $a_i$ to that part of $G^*_1$ containing $H_2$, and $D_1$ contains but a part of these arcs. Hence $a_i$ is still joined to the rest of this part of $G^*_1$, and $G^*_2$ contains the same number, 2, of connected pieces as $G^*_1$, a contradiction. This proves the statement.

Let us interchange the names of the chains $A'_1(H_1)$, $A'_2(H_1), \cdots, A'_n(H_1)$ so that they occur in that order in $P'$, and rename the vertices $a_1, a_2, \cdots, a_n$ and their arcs accordingly. Let $a'_1$ be the vertex joining $A'_1(H_1)$ and $A'_2(H_1), \cdots$, let $a'_{n-1}$ be the vertex joining $A'_{n-1}(H_1)$ and $A'_n(H_1)$, and let $a'_n$ be the vertex joining $A'_n(H_1)$ and $A'_1(H_1)$. These vertices form a *corresponding set of vertices* to the vertices $a_1, \cdots, a_n$. If we map the graphs together on a sphere ‡ as described in N, theorem 30, we can draw a closed curve which passes successively through the vertices $a_1$, $a'_1$, $a_1$, $a'_2$, $\cdots$, $a_n$, $a'_n$, and touches no other vertices or arcs. We shall show that $a'_1, \cdots, a'_n$, if dropped out, disconnect $G'$.

Form $G'_{n-1}$ from $G'$ by letting $a'_n$ and $a'_{n-1}$ coalesce. Form $G_{n-1}$ from $G$

† The resulting graph is in just two pieces as $a_i$ is not a cut vertex of $G$, $G$ being non-separable.

‡ So that each arc of one graph cuts the corresponding arc of the other, and otherwise the graphs do not intersect.

by replacing the vertex $a_n$ by the two vertices $a_n(H_1)$ and $a_n(H_2)$, letting the arcs $A_n(H_1)$ end on $a_n(H_1)$, and letting the arcs $A_n(H_2)$ and $A_n(A)$ end on $a_n(H_2)$. By theorem 8, $G_{n-1}$ and $G'_{n-1}$ are duals. Form $G'_{n-2}$ from $G'_{n-1}$ by letting $a'_{n-1}$ and $a'_{n-2}$ coalesce. Form $G_{n-2}$ from $G_{n-1}$ by replacing the vertex $a_{n-1}$ by the two vertices $a_{n-1}(H_1)$ and $a_{n-1}(H_2)$, and replacing the arcs on $a_{n-1}$ as above.* Then $G_{n-2}$ and $G'_{n-2}$ are duals. Continuing, we have finally the dual graphs $G_1$ and $G'_1$, where $a'_1, a'_2, \cdots, a'_n$ have all coalesced to form a single vertex in $G'_1$, and $a_i$ has been replaced by the two vertices $a_i(H_1)$, $a_i(H_2)$, $i = 2, 3, \cdots, n$, in $G_1$.

Let $I_1$ be $H_1$ together with the vertices $a_1$, $a_2(H_1), \cdots, a_n(H_1)$, and the arcs $A_1(H_1), \cdots, A_n(H_1)$. Define $I_2$ similarly. Then $I_1$ and $I_2$ have but the vertex $a_1$ in common, and together they form $G_1$. Let $I_{11}, \cdots, I_{1k}$ be the components of $I_1$, (there may be but one), and let $I_{21}, \cdots, I_{2l}$ be those of $I_2$. $I_1$ is of nullity $> 0$. For if $H_1$ is of nullity $> 0$, $I_1$ is, as it contains $H_1$. Otherwise, by hypothesis, there are at least two arcs in one of the sets of arcs $A_i(H_1)$. Let these be the arcs $a_ib$ and $a_ic$ in $G$. $b$ and $c$ are joined by a chain in $H_1$, as $H_1$ is connected; this chain together with the arcs $a_ib$ and $a_ic$ forms a circuit in $I_1$. The statement now follows from N, theorem 4. Similarly, $I_2$ is of nullity $> 0$. Thus one of the components $I_{1i}$ of $I_1$, and one of the components $I_{2j}$ of $I_2$, is of nullity $> 0$ by N, theorem 13.†

Let $I'_{11}, \cdots, I'_{2l}$ be the subgraphs of $G'_1$ whose arcs correspond to the arcs of $I_{11}, \cdots, I_{2l}$. By N, theorem 25,‡ $I'_{12}, \cdots, I'_{2l}$ are the components of $G'_1$, and $I'_{11}$ is a dual of $I_{11}, \cdots, I'_{2l}$ is a dual of $I_{2l}$. As $I_{1i}$ and $I_{2j}$ are of nullity $> 0$, $I'_{1i}$ and $I'_{2j}$ are of rank $> 0$, and they contain therefore no 1-circuits. Drop out any 1-circuits there may be in $G'_1$. The resulting graph $J'$ is separable, as it contains both $I'_{1i}$ and $I'_{2j}$. As it is connected, it has a cut vertex $x'$. $J'$ is formed therefore of two graphs $J'_1$ and $J'_2$, each containing an arc and hence at least two vertices, which have only the vertex $x'$ in common. If $x'$ is dropped out of $J'$, $J'$ is disconnected. Hence if $x'$ is dropped out of $G'_1$, forming the graph $K'_1$, $K'_1$ is not connected.

$x'$ is the vertex $a'_1$. For suppose it were not. Form $K'$ from $G'$ by dropping out $x'$ and its arcs. As $G$ is doubly connected, we see easily that $G'$ is also, by N, theorem 26,§ and hence $K'$ is connected. $K'_1$ is formed from $K'$ by letting the vertices $a'_1, \cdots, a'_n$ coalesce, and hence $K'_1$ also is con-

* It is easily seen that $a_{n-1}$ is not a cut vertex of $G_{n-1}$, and hence $A'_{n-1}$ is still a circuit in $G'_{n-1}$.

† P, theorem 3.

‡ P, theorem 13.

§ P, theorem 14.

nected, a contradiction. Now drop out the vertices $a'_1, \cdots, a'_n$ and their arcs from $G'$. We are left with the graph $K'_1$ which is not connected, and the theorem is proved.

THEOREM 10. *A dual $G'$ of a triply connected graph $G$ containing no 1- or 2-circuits is a similar graph.*

$G'$ contains no 1- or 2-circuits; for if it did, dropping out the corresponding one or two arcs of $G$ would disconnect this graph, contrary to theorem 5.

$G'$ contains at least four vertices. For if it contained less than four vertices, it would contain at most three arcs, and would be of nullity $\leqq 1$, and $G$ would be of rank $\leqq 1$, and would contain but two vertices.

Finally, $G'$ cannot be disconnected by dropping out but two vertices. For suppose it could. If dropping out $a'_1$ and $a'_2$ and their arcs leaves the two parts $H'_1$ and $H'_2$, then, as $G$ cannot be similarly disconnected, either $a'_1$ and $a'_2$ are each joined to $H'_1$ by but a single arc, or they are each joined to $H'_2$ by but a single arc, say the first, by theorem 9. The two arcs joining $a'_1$ and $a'_2$ to $H'_1$, if dropped out, disconnect $G'$. Hence the corresponding two arcs of $G$ form a circuit. But this is contrary to the hypothesis that $G$ contains no 2-circuit. Thus $G'$ is triply connected.

THEOREM 11. *A triply connected planar graph containing no 1- or 2-circuit has a unique dual.*

For suppose $G'$ and $G''$ were both duals of the triply connected graph $G$. Let $H'$ be any subgraph of $G'$ of nullity 1. Let $H''$ be the corresponding subgraph of $G''$ (the correspondence being given through the graph $G$), and let $H$ be the complement of the corresponding subgraph of $G$. Then

$$r = R - n',$$
$$r = R - n'',$$

and hence

$$n'' = n',$$

and $H''$ is of nullity 1. Similarly, if $H''$ is any subgraph of $G''$ of nullity 1, the corresponding subgraph $H'$ of $G'$ is of nullity 1. Moreover, by theorem 10, $G'$ and $G''$ are both triply connected, and neither contains any 1- or 2-circuits. Therefore, by theorem 3, $G'$ and $G''$ are congruent; that is, the dual of $G$ is unique.

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# ON THE DECOMPOSABILITY OF CLOSED SETS INTO A COUNTABLE NUMBER OF SIMPLE SETS OF VARIOUS TYPES.

By G. T. WHYBURN.

1. In this paper we shall consider the following general problem: *Given any class $K$ of closed point sets, under what conditions is a compact metric space $C$ expressible as the sum of a countable number of $K$-sets?*

It is obvious that this problem is a much more general formulation of the well known problem * in the theory of curves of finding necessary and sufficient conditions in order that a given curve should be the sum of a countable number of arcs. For, in order to obtain the solution to this special problem from the solution to our general problem we have only to take $K$ to be the class of simple arcs and take $C$ to be the curve. This particular problem was considered by Menger in 1926, who formulated it for *regular curves* and gave the solution for the special case of acyclic curves. The same problem in slightly more general form was considered by the present author in 1929 † and was reduced, for the case of any locally connected continuum, to the same problem concerning the cyclic elements of that continuum. This reduction gave at once the solution to the problem in a number of special cases including those solved by Menger. These remarks suffice to show the point of view from which this problem has been attacked, i. e., the already known classes of continua have been examined in turn to determine if they do or do not have the property of being expressible as the sum of a countable number of arcs or to find conditions under which they do have this property. Since it was found that certain curves of a particular type do have this property whereas other curves of this same type and of apparently no greater complexity do not have this property, the possibility of a general solution to the problem seemed remote. However if, as we shall do in this paper, we concede at once that the continua which have the property of being the sum of a countable number of arcs (or, in general, of $K$-sets) form a distinct class by themselves which is not to be identified with any of the known types

* See, for example, K. Menger, *Mathematische Annalen*, Vol. 96 (1926), pp. 575-582, and *Jahresbericht der Deutschen Mathematischer-Vereinigung*, Vol. 35 (1926), p. 146.

† See *Fundamenta Mathematicae*, Vol. 14 (1929), pp. 103-106.

in our present classification of curves, and question ourselves as to what type of complexity one can get in a continuum which is formed by adding together two, three, $n$, or a countably infinite number of arcs (or, in general, of $K$-sets) we are led at once to those properties which are characteristic for the decomposability of any closed set into a countable number of arcs or of $K$-sets. Thus we find the solution not only to the arc-problem but to the general problem for any class $K$ of closed sets.

There are many other interesting cases (i. e., choices of the class $K$) to our general problem and its solution to be given below in §§ 3 and 4 besides that in which $K$ is the class of simple arcs. Some of these will be discussed below in § 5. Notably, for a reason to be mentioned later, the case where $K$ is the class of acyclic curves is of especial interest and importance. Also closed sets which are the sum of *finite numbers* $2, 3, \cdots n, \cdots$ of $K$-sets have interesting properties which will be discussed in § 6.

Before proceeding with the solution to our problem we introduce a new notion defining sets which may be regarded as

2. *Generalized derived aggregates.* Let $C$ be any compact metric space, let $A$ be any closed subset of $C$ and let $K$ be any class of closed sets. Then by the $K$-derivative $K(A)$ of $A$ is meant the set of all points $x$ of $A$ such that for no neighborhood $U$ of $x$ is $A \cdot U$ contained in any $K$-set in $C$. Set

$$\begin{aligned} A_K^1 &= K(A), \\ A_K^2 &= K(A_K^1), \\ &\cdots\cdots\cdots \\ A_K^n &= K(A_K^{n-1}), \\ &\cdots\cdots\cdots \end{aligned}$$

in general, $A_K^\alpha = K(A_K^{\alpha-1})$ or $= \prod_{\beta<\alpha} K(A_K^\beta)$, according as the ordinal number $\alpha$ does or does not have an immediate predecessor.

For example, if we take $K$ to be the class of all single points, then for any closed set $A$, the $K$-derivatives of $A$ are equal respectively to the ordinary successive derived aggregates of the set $A$, i. e., $A_K{}^\alpha = A^{(\alpha)}$ for every $\alpha$. Other examples are as follows. Let $H$, $K$, and $L$ denote respectively the classes of simple arcs, simple closed curves, and acyclic curves. Then if $C$ is a circle plus two diameters $xoy$ and $poq$, $C_H{}^1 = x + y + o + p + q$ while $C_K{}^1 = C_L{}^1 = C_H{}^2 = 0$. If $C$ is the curve obtained by adding to a rectangle $abcd$ a sequence of its altitudes, $a_1b_1$, $a_2b_2$, $a_3b_3$, $\cdots$ converging to $ab$, then $C_H{}^1 = ab + (a_1 + b_1) + (a_2 + b_2) + \cdots$, $C_H{}^2 = 0$, $C_K{}^1 = C_L{}^1 = ab$, and $C_K{}^2 = C_L{}^2 = 0$. If we add on to this curve a sequence of altitudes con-

verging to each of the altitudes $a_ib_i$, clearly we get a curve whose third but not second $H$-, $K$-, and $L$-derivatives are vacuous. These examples perhaps suffice to indicate how the $K$-derivatives may be used to describe or characterize various stages of complexity in the structure of a continuum.

It follows at once from the definition that all the $K$-derivatives of a closed set $A$ are themselves closed sets and that they decrease monotonically with successive derivations, i. e., $A_K{}^\alpha \supset A_K{}^{\alpha+1}$.

3. THEOREM. *In order that the compact metric space $C$ should be the sum of a countable number of $K$-sets, where $K$ is any given class of closed point sets, it is necessary and sufficient that $C_K{}^\beta = 0$ for some ordinal number $\beta$ of the first or second class.*

To prove the sufficiency of the condition, let us suppose that $C_K{}^\beta = 0$, where $\beta$ is an ordinal number of the first or second class. For symmetry let us set $C = C_K{}^0$. Now each point $x$ of $C - C_K{}^1$ is contained in a neighborhood $Q_x$ such that some $K$-set $K_x$ in $C$ contains $C \cdot Q_x$. By the Lindelöf Theorem, a countable collection $[Q_i{}^0]$ of the neighborhoods $[Q_x]$ covers $C - C_K{}^1$, and thus $C - C_K{}^1$ is contained in the sum $G_0$ of the elements of the corresponding countable collection of $K$-sets $[K_i{}^0]$. Likewise $C_K{}^1 - C_K{}^2$ is contained in the sum $G_1$ of a countable number of $K$-sets, $K_1{}^1, K_2{}^1, K_3{}^1, \cdots$ in $C$. In general, for each $\alpha < \beta$, $C_K{}^\alpha - C_K{}^{\alpha+1}$ is contained in the sum $G_\alpha$ of a countable number of $K$-sets $K_1{}^\alpha, K_2{}^\alpha, \cdots$ in $C$. But it is seen at once that

$$C = \sum_{0 \le \alpha \le \beta} (C_K{}^\alpha - C_K{}^{\alpha+1}) \subset \sum_{0 \le \alpha \le \beta} G_\alpha = \sum_{0 \le \alpha \le \beta} \sum_{n=1}^{\infty} K_n{}^\alpha$$

and, there being only a countable number of ordinals $\alpha < \beta$, we thus have expressed $C$ as the sum of a countable number of $K$-sets.

The condition is also necessary. For suppose $C = K_1 + K_2 + K_3 + \cdots$, where each $K_i$ is a set of class $K$, and suppose, contrary to our theorem, that $C_K{}^\alpha$ is non-vacuous for all $\alpha$'s of the first and second classes. Then since there are uncountably many of the sets $C_K{}^\alpha$, it follows by the well known theorem that (in the space we are considering) any well-ordered monotone decreasing sequence of distinct closed sets is countable that for some ordinal $\beta$ of the first or second class, $C_K{}^\beta = C_K{}^{\beta+1}$. In other words, for some $\beta$, $K(C_K{}^\beta) = C_K{}^\beta$ and hence $C_K{}^\beta$ is its own $K$-derivative. Now clearly there exists a point $x_1$ of $C_K{}^\beta$ which does not belong to $K_1$; and since $K_1$ is closed, a neighborhood $R_1$ of $x_1$ exists such that $\bar{R}_1 \cdot K_1 = 0$. Likewise, since, by virtue of the definition of $K$-derivative, $R_1 \cdot C_K{}^\beta$ cannot be contained in any $K$-set in $C$, there exists a point $x_2$ in $R_1 \cdot C_K{}^\beta$ not belonging

to $K_2$. There exists a neighborhood $R_2$ of $x_2$ such that $\bar{R}_2 \subset R_1$ and $\bar{R}_2 \cdot K_2 = 0$. Similarly in $R_2 \cdot C_K^\beta$ there exists a point $x_3$ not belonging to $K_3$ and there exists a neighborhood $R_3$ of $x_3$ such that $\bar{R}_3 \subset R_2$ and $\bar{R}_3 \cdot K_3 = 0$, and so on. Continuing in this way we obtain a sequence of neighborhoods $R_1, R_2, R_3, \cdots$ in $C$ such that for each $n$, $\bar{R}_n \subset R_{n-1}$ and $\bar{R}_n \cdot \sum_1^n K_i = 0$. But then $\prod_1^\infty \bar{R}_i \cdot \sum_1^\infty K_i = \prod_1^\infty \bar{R}_i \cdot C = 0$, contrary to the fact that $\prod_1^\infty \bar{R}_i$ is not vacuous and is a subset of $C$. This contradiction proves our theorem.

4. *An alternate form.* We can state our proposition just proved in the following equivalent form.

*In order that a compact metric space $C$ should be the sum of a countable number of $K$-sets it is necessary and sufficient that each subset $N$ of $C$ contain at least one point $x$ for some neighborhood $R$ of which it is true that $N \cdot R$ is contained in some $K$-set in $C$.*

For if this condition is satisfied, then by choosing $N$ to be the successive $K$-derivatives of $C$ we see that these sets $C_K^\alpha$ must be distinct sets and since they are well-ordered and monotone decreasing, they must be countable in number. Thus they must be vacuous after a certain $\beta$ of the first or second class, and by our theorem in § 3, $C$ is the sum of a countable number of $K$-sets. If, on the other hand, $C$ is the sum of a countable number of $K$-sets, then $C_K^\beta = 0$ for some $\beta$ of the first or second class; but it is easily seen that if any subset $N$ of $C$ failed to contain a point $x$ such that the part of $N$ (18) some neighborhood of $x$ is contained in a $K$-set in $C$, then $N$ would belong to every set $C_K^\beta$ for all ordinal numbers $\beta$.

5. *Applications. Some particular choices for $K$.*

(a). In general, for any given class $K$, let us call a continuum (= a self-compact, connected, metric space) $C$ which is the sum of a countable number of $K$-sets a *$K$-sum*. Then from § 4 it follows at once that *in any $K$-sum $C$ there exists an everywhere dense set of open sets each of which is contained in some $K$-set in $C$.*

(b). If we take $K$ to be the class of simple arcs, then § 4 tells us that in order that a continuum $C$ should be an arc-sum it is necessary and sufficient that every subset $N$ of $C$ contain a point $x$ for some neighborhood $R$ of which it is true that $N \cdot R$ is a subset of some arc in $C$. Thus we have a solution to the arc-sum problem. In particular, if we take $N = C$, we see that any arc-sum $C$ must contain open sets which are contained in arcs in $C$

and thus it must contain "free-arcs," i. e., arcs $ab$ such that no point of $ab - (a + b)$ is a limit point of $C - ab$. Indeed, by (a) it follows that *in any arc-sum the free arcs must be everywhere dense.* Suppose we call a continuum in which the free arcs are everywhere dense a *free arc continuum.* Then we have that *any arc-sum is a free arc continuum.*

We note here, in answer to a problem proposed by Menger (*loc. cit.*), the fact that *any arc-sum $A$ is the sum of the elements in a null sequence of arcs,* i. e., $A = \sum_1^\infty B_n$, where each $B_n$ is an arc and $\delta(B_n) \to 0$ as $n \to \infty$. Clearly this is the case; for if $A = \sum_1^\infty A_n$, where each $A_n$ is an arc, then for each $n$, $A_n$ is the sum of a finite number of arcs $A_n^i$ each of diameter $< 1/n$; and obviously we can arrange these arcs $[A_n^i]$ into a null sequence. Indeed, similar reasoning yields at once the result that *if $K$ is any class of closed sets such that each $K$-set is itself the sum, for each $\epsilon > 0$, of a finite number or of a null sequence of $K$-sets of diameter $< \epsilon$, then every $K$-sum is the sum of the elements in a null sequence of $K$-sets.*

(c). Suppose now we take $K$ to be the class of acyclic curves, i. e., the locally connected continua containing no simple closed curves. Then of course §§ 3 and 4 give us necessary and sufficient conditions for a continuum $C$ to be an acyclic curve-sum. Now in case $C$ itself is a locally connected continuum, this choice of $K$ is of special significance, because of the fact * that any closed subset of such a set $C$ which is homeomorphic with a subset of some acyclic curve *is really contained in an acyclic curve in $C$.* Thus in this case we can characterize acyclic curve-sums completely in terms of topological properties of their subsets as follows: *A locally connected continuum $C$ is an acyclic curve-sum if and only if every closed subset $N$ of $C$ contains a point $x$ such that some closed neighborood of $x$ in $N$ is homeomorphic with a subset of some acyclic curve* (i. e., has only point- and acyclic curve-components and only a null sequence of components which are not points). Also in this case it follows at once that *any locally connected subcontinuum of an acyclic curve-sum is itself an acyclic curve-sum.* It has been remarked by Menger (*loc. cit.*) that the corresponding proposition about arc-sums is not valid. Thus it appears that in many respects a detailed study of acyclic-curve-sums would be more fruitful than one of arc-sums.

(d). If we choose $K$ to be the class of free arc continua, we get at once

* This is established for $C$ in the plane by H. M. Gehman [See *Transactions of the American Mathematical Society,* Vol. 29 (1927), p. 560], and has been extended to general metric locally connected continua by L. Zippin.

from § 4 or from (a) that the free arcs are dense in any free arc continuum-sum and hence *any free arc continuum-sum is itself a free arc continuum.*

(e). As a final application, let us choose $K$ to be the class of continua without continua of condensation, which we shall call $w$-continua. Then since any $w$-continuum is * a *free arc continuum,* it follows as a corollary to (d) that any $w$-continuum-sum is a free arc continuum. Furthermore it is readily proved with the aid of some elementary results of dimension theory that *any subset of a w-continuum-sum which contains no arc is of dimension zero.* Thus any 1-dimensional subset (e. g., any connected subset) of a $w$-continuum-sum must contain an arc. In particular (since an arc is a $w$-continuum), any arc-sum has the property just mentioned.

6. *The finite cases.* In conclusion we make some remarks concerning the cases in which $C_K^n = 0$ for finite integers $n$. It might be supposed, by analogy with our result in § 3, that the vanishing of $C_K^n$ for a finite integer $n$ comprises a necessary and sufficient condition that $C$ be the sum of $n$ $K$-sets. However, we shall see at once that in general this is not the case. That the condition is not sufficient is seen for example, in the continuum $C$ obtained by adding on to the unit interval a perpendicular ordinate of length $1/q$ for each point $p/q$ where $p$ and $q$ are integers prime to each other and $p < q$; for in this case, clearly $C$ is not the sum of any finite number of arcs, whereas its second arc-derivative vanishes (i. e., $C_K^2 = 0$, where $K$ is the class of simple arcs). It will be shown below, however, that this condition is necessary.

In general, *if $C$ is the sum of a finite number of $K$-sets, then for any integer $n$, every point $C_K^n$ belongs to at least $n+1$ of these sets.* For clearly every point of $C_K^1$ belongs to at least 2 of the $K$-sets; and assuming that for an integer $k-1$, every point of $C_K^{k-1}$ belongs to at least $k$ of the sets, it follows that every point of $C_K^k$ belongs to at least $k+1$. For if not, then some point $x$ of $C_K^k$ belongs to at most $k$ of the sets, say to $K_1, K_2, \cdots, K_k$; hence some neighborhood $U$ of $x$ in $C$ is a subset of $K_1 + K_2 + \cdots + K_k$ and thus $C_K^{k-1} \cdot U \subset K_1 + K_2 + \cdots + K_k$; but then since by supposition every point of $C_K^{k-1}$ belongs to at least $k$ of the sets, it follows that $C_K^{k-1} \cdot U \subset K_1 \cdot K_2 \cdots K_k$; thus the part of $C_K^{k-1}$ in a neighborhood of $x$ is a subset of $K$-set in $C$, (e. g., of $K_1$), contrary to the fact that $x \subset C_K^k$. Therefore, by induction, it follows that for any $n$, every point of $C_K^n$ belongs to at least $n+1$ of the sets.

Now, in particular, if $C$ is the sum of $n$ $K$-sets, every point of $C_K^{n-1}$

* See Urysohn, *Verhandelingen der Akademie te Amsterdam,* Vol. 13 (1927), No. 4.

is contained in all $n$ of these sets and hence $C_K^{n-1}$ is contained in their product. It follows from this that $C_K^n = 0$, because $C_K^{n-1}$ is a subset of some single $K$-set in $C$. This also follows because if $C_K^n$ contained a point $x$, then $x$ would have to belong to $n+1$ of the sets, whereas there are only $n$ sets in all. Therefore we have proved that *In order that a set $C$ be the sum of $n$ $K$-sets it is necessary that $C_K^{n-1}$ be a subset of a $K$-set in $C$ and that $C_K^n = 0$.*

Although, as we have seen, this condition is not sufficient for classes $K$ in general, nevertheless it seems possible that it may be sufficient for certain particular classes $K$. Notably when $K$ is the class of acyclic curves does this seem possibly to be the case. It would be interesting to study the finite cases further and find necessary and sufficient conditions that a set $C$ be the sum of $n$ $K$-sets in general and for particular choices of $K$.

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# GENERALIZATION OF A THEOREM DUE TO C. M. CLEVELAND.†

By LEO ZIPPIN.‡

Recently, C. M. Cleveland has announced the theorem that: if, in the plane, $M$ is a bounded continuous curve which contains no domain and $K$ is a totally disconnected closed subset of $M$ such that $M - K$ is connected, there exists an acyclic continuous curve $T$ containing $K$ and such that 1) all endpoints of $T$ belong to $K$, 2) the point set $M \cdot T$ is disconnected, 3) $M - M \cdot T$ is connected.§ It is, of course, clearly understood that $T$ belongs to the plane of $M$. It was the suggestion of Professor R. L. Moore to extend this result to euclidean spaces of higher dimension. It is to be expected that the theorem will admit of considerable elaboration in higher spaces, and we give two distinct versions both considerably stronger than the original form; however, the methods of this paper do not admit of application in the plane case, and the results are not generally true in the plane. Our theorems generalize, also, certain results of R. L. Moore relating to "paths that do not separate a given continuous curve," and results due to Zarankiewicz. The arguments of this paper rest principally on well known theorems of Dimension-Theory, and the local arcwise connectedness of generalized continuous curves $C$, i. e., complete separable metric spaces, connected and locally connected.¶ We give, in particular, a necessary and sufficient condition that a self compact totally disconnected subset of $C$ be the set of endpoints of an acyclic continuous subcurve of $C$.

*Preliminary:* Let $B$ be a closed, non-dense subset of a generalized continuous curve $C$ (in the sense above) which does not locally separate $C$.‖

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† Presented to the American Mathematical Society, February 11 (1930). Read at the Easter Meeting, Berkeley, California.

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§ See, *Bulletin of the American Mathematical Society*, Abstract, Vol. 36, p. 331. The essential condition that $M - K$ is connected is, by oversight certainly, not there stated.

¶ See R. L. Moore, *Bulletin of the American Mathematical Society*, Abstract, Vol. 33 (1927), p. 141; Synopsis of Boulder Colloquium Lectures (August 1929); the Colloquium Lecture to appear in the regular series. Also, K. Menger, *Monatshefte für Mathematik und Physik*, Vol. 36 (1929), p. 212; N. Aronszajn, *Fundamenta Mathematicae*, Vol. 15 (1930), p. 228; in the same volume, C. Kuratowski, p. 301.

‖ We shall say that a subset $V$ of $C$ locally separates $C$ provided there exists an open connected subset $U$ of $C$ such that $U \cdot V = V'' \subset \bar{V}'' \subset U$, and $U - V''$ is not connected.

There exists a countable set $P$ of points of $C - B$, dense in $C - B$ and therefore dense in $C$. For every pair $p$ and $q$ of points of $P$ there is an arc $pq$ in $C - B$: for $C - B$ is by hypothesis connected, and is connected im kleinen since $B$ is closed; and being open in $C$, it is by a theorem due to Alexandroff † in itself a complete metric space.

We suppose well ordered the denumerable set of pairs $p$ and $q$ of points of $P$. For each pair $(p, q)$ let $pq$ be an arbitrary arc of $C - B$ of diameter less than $X_{pq} + 1/n$, where $n$ is the integer associated with the pair $(p, q)$ in the well ordering, and $X_{pq}$ is the lower limit of the set of diameters of the totality of arcs joining $p$ and $q$ in $C - B$. Let $K$ represent the point set which is the sum of all of these arcs.

It is clear that $K \cdot B = 0$, that $K$ is connected, and since $B$ is closed, that $K$ is connected im kleinen: for the last, if $k$ is a point of $K$, there is a $d$ such that no point of $C$ at a distance less than or equal to $d$ from $k$ belongs to $B$; in symbols, $\bar{S}(k, d) \cdot B = 0$. Let $n$ be an integer such that $1/n < d/2$. There is an $e$ such that two points of $S(k, e)$ may be joined by an arc of $S(k, 1/n)$ and such that $S(k, e)$ contains no pair of points of $P$ of subscript less than or equal to $n$. Then $K \cdot S(k, e)$ belongs to a connected subset of $S(k, d)$. If $H$ is any subset of $C - K$ we shall show that $H$ does not even locally separate $C$. Otherwise there is an open connected subset $U$ of $C$ such that $U \cdot H = H'' \subset \bar{H}'' \subset U$, and $U - H''$ is not connected. Then $H''$ contains a closed subset $H^*$ which irreducibly separates $U$ between two points of $U$.‡ It follows from our condition on $B$ that $H^*$ is not a subset of $B$ and contains a point $h$ not of $B$. There is an $e$ such that $S(h, e) \subset U - B \cdot U$ and a $d$ such that two points of $S(h, d)$ may be arc-joined in $S(h, e/2)$. Let $n$ be an integer such that $1/n < e/2$. There exist two sequences of points of $P$, $(p_i)$ and $(q_i)$ belonging to distinct components of $U - H^*$, and converging to $h$. There exists a pair of points $p_j$ and $q_k$ such that $p_j + q_k \subset S(h, d)$ and the point pair $| (p_j, q_k) |$ is of subscript greater than $n$ in the well ordering of pairs of points of $P$. The arc $p_j q_k$ of $K$ belongs to $U - H^*$.

With the weaker condition on $B$ that $B$ does not separate $C$ (and other-

† "Sur les ensembles de la première classe et les ensembles abstraits," *Comptes Rendus*, Vol. 178 (1924), pp. 185-187. (That the metric which exhibits the completeness of $C - B$ is in general different from the metric originally associated with $C$ need not detain us, who are concerned merely with the fact of its arcwise connectedness).

‡ Compare G. T. Whyburn, "Irreducible Cuttings of Continua," *Fundamenta Mathematicae*, Vol. 13 (1929), p. 49, Theorem 8: the thought of this proof generalizes readily.

wise $K$ as defined above is not itself connected), it is immediate from the connectedness and "everywhere-density" of $K$ that no subset of $C - K$ separates $C$. With the stronger condition on $B$ that $B$ does not even locally separate in the weak sense, i. e. if $U$ is an open connected subset of $C$ and $U \cdot B = B^*$ then $U - B^*$ is connected, it follows as above that no subset of $C - K$ locally separates even in the weak sense. It may be of interest to note that for the case that $B$ is vacuous, $K$ defined above includes all cut points and local cut points of $C$, and that this fact may be employed to yield simple proofs that these sets are at most one dimensional and are $F_\sigma$-subsets of $C$.†

In view of a recent paper by Ayres which has just been received by the author, it has seemed well to interpolate here a justification of the previous statements, especially since the lemma ‡ there employed is not valid for generalised continuous curves: witness the simple example that $C$ is the following point set (in the plane): $0 \leq x \leq 1$, $y = 0$; $0 \leq y \leq 1$, $x = 1/n$, $(n = 1, 2, 3, \cdots)$. Here the point $(0, 0)$ is not a cut-point, while the points $(1/n, 0)$ belong to $H_1$ in Ayres' notation. Now, if the point $y$ of $C$ is a local cut-point of $C$ there is an $e$ such that $y$ is a cut-point of the composant $U_y$ of $S(y, e)$. If $y$ is a point of an arc $pq$ of $K$, defined above, we may choose $e$ so that $p$ and $q$ are not in $S(y, e)$. On $pq$ there is a pair of points $x$ and $z$, in order $pxyzq$, such that $(yx - x) + (yz - z) \subset U_y$, and $x + z$ do not belong to $U_y$. If $y$ fails to separate some point of $yx - x$ from some point of $yz - z$ it cannot separate any pair of points of these respective sets, and there is a point $u$ of $U_y$ separated by $y$ from both of these point sets: in this case there is an arc $uy$ which has only $y$ in common with $pxyzq$, and the set of such points $y$ (by a theorem of Zarankiewicz, from the separability and metricity of $C$) is at most a countable set $Y$. If the point $y$ is not a point of $Y$ it must separate points of $yx - x$ from points of $yz - z$ (in $U_y$ for the case of local cut-points) and belongs to $H_n$ where $1/n \leqq e$.§

Suppose now that $y'$ is not a point of $H_n$, for a given $n$, and is not a point of $Y$, but is a point of the arc $pq$ (with the understanding as before that $p$ and $q$ are not in $S(y', 1/n)$, the other case offering no difficulty, and

† G. T. Whyburn (and references there to Zarankiewicz), "Concerning Points of Continuous Curves Defined by Certain im kleinen Properties," *Mathematische Annalen*, Vol. 102 (1929), p. 318, Theorem 8.

‡ W. L. Ayres, "A New Proof of a Theorem of Zarankiewicz," *Fundamenta Mathematicae*, Vol. 16 (1930), p. 134. Compare, for this section, G. T. Whyburn, "On the Structure of Connected and Connected im kleinen Point sets," *Transactions of the American Mathematical Society*, Vol. 32, pp. 926-928.

§ It will be clear that we are defining $H_n$.

the endpoints $p$ and $q$ being considered separately from the inner points). There is then an arc $x'y'z'$ of $pq$, such that $x'y'z'$ belongs to $S(y', 1/n)$ excepting for its endpoints. Since $y'$ is not in $H_n$ there is an arc in $U_{y'}$, the component containing $y'$ of $S(y', 1/n)$ joining the point $x''$ of $x'y' - y'$ to a point $z''$ of $z'y' - y'$, the arc having these points only in common with $x'y'z'$, and being contained in $S(y', d)$ for some positive number $d < 1/n$. It is readily seen that there is a neighborhood of $y'$ no point $y^*$ of which separates points of the corresponding are $x^*y^*z^*$ in the component $U_{y^*}$ of $S(y^*, 1/n)$, since for points sufficiently near to $y'$, $x^*y^*z^* \supset x'y'z'$ and $U_{y^*} \supset U^0_{y''}$, the component of $y'$ in $S(y', d)$: in other words, $H_n$ defined for $pq$ is closed there. Then the set of local cutpoints of $C$, these being contained in arcs of $K$, is the sum of a countable number of point sets, and each of these is the sum of a countable number of closed sets plus a countable point set $Y$. Finally we add those points of $P$, end-points of arcs $pq$, which are local cut-points. The local cut-points of $C$ form an $F_\sigma$-subset of $C$: and an entirely similar argument holds for the set of cut-points. Finally that these sets are at most one-dimensional, is an immediate consequence of their inclusion in a one-dimensional point set.

We turn to the first generalization of Cleveland's Theorem.

THEOREM 1. *If $C$ is a continuous curve, lying in $E_n$ $(n > 2)$, a euclidean space of n-dimensions, and contains no domain in $E_n$, and $B$ is a closed and totally disconnected subset of $E_n$ such that $B'' = B \cdot C$ does not separate (locally separate) $C$, then there exists a ray $L$ in $E_n$ such that* 1) $L \supset B$, 2) $L \cdot C = B^0$ *is totally disconnected,* 3) $B^0$ *does not separate (locally separate) $C$,* 4) $B^0 - B \subset \bar{C}^{n-1}$, *where $C^{n-1}$ denotes the points of $C$ which are of dimension at least $n-1$ in $C$,*† 5) *the endpoint of $L$ is an arbitrary point of $B$,* 6) *if $B$ is bounded, $L$ is an arc with arbitrary endpoints in $B$.*

There exists a $G_\delta$-subset $G$ of $C$ such that $G$ is of dimension zero, $C - G$ is an $F_\sigma$ of dimension at most $n-2$, and every point of $G$ is a point of $\bar{C}^{n-1}$.‡ From our preliminary considerations there exists in $C$ a one-dimensional $F_\sigma$-subset $K$, such that no subset of $C - K$ separates (locally separates) $C$. The set $K + (C - G)$ is an $F_\sigma$-subset of $C$ of dimension $n-2$ at most.§ The complement in $C$ of $K + (C - G)$ is a subset $G''$

† Dimension is to be understood throughout in the Menger-Urysohn sense.

‡ This follows from a slight modification of the argument given by Menger, *Dimensiontheorie*, p. 110, Teubner (1928).

§ *Ibid.*, p. 93.

of $G$. The complement in $E_n$ of $K + (C - G)$ is a $G_\delta$-subset $M''$. It is readily seen that $M'' + B$ is a $G_\delta$-subset $M$ of $E_n$.

That there is in $E_n$ a ray $L''$ which contains $B$ is not difficult to establish; it is most easily deduced from an interesting theorem due to G. T. Whyburn,† as follows. If $B$ is unbounded, we invert $E_n$ with respect to a center of inversion $v$ not belonging to $B$. Then $v + B^*$, the image of $B$, is bounded, closed, totally disconnected. There is an arc $x^*v$ in $E_n$, where $x^*$ is an arbitrary point of $B^*$, containing $B^* + v$. The image of $x^*v$ in $E_n$ is a ray $L''$ whose endpoint is $x$, the image of $x^*$. In the case that $B$ is bounded, the theorem of Whyburn gives immediately an arc $xy$ in $E_n$ containing $B$, and the argument of the sequel may be applied directly to it. Since $B$ is closed and totally disconnected, the complement of $B$ in $L''$ is a countable set of intervals with endpoints $a$ and $b$ in $B$. Let $ab$ be such an interval. Let $e$ be an arbitrary positive number less than 1. It is readily seen that there exists an infinite chain of spheres of $E_n$ of order type $w^* + w$, i. e., $\cdots S_{-n}, S_{-(n-1)}, \cdots S_0, S_1, \cdots S_n, \cdots$, such that the diameter of $S_j$ is at most the numerical value of $e/j$ (at most $e$, if $j = 0$), $\bar{S}_i$ and $\bar{S}_j$ have a common point if and only if $i$ and $j$ are consecutive, $S_i \cdot ab$ is not vacuous for any $i$, and $S_i \cdot (L'' - ab) = 0$ for every $i$. Since the complement of $M$ in $E_n$ is of dimension at most $n - 2$, $M$ is everywhere dense in $E_n$ and there exists a chain of points of $M$ of order type $w^* \neq w$, $\cdots x_{-n}, \cdots x_0, \cdots, x_n, \cdots$ such that $x_0 \subset S_0$, and $x_i \subset S_i \cdot S_{i+1}$ if $i$ is positive, and $x_i \subset S_i \cdot S_{i-1}$ if $i$ is negative.

Now let $x$ be any point of $M$ and $S$ any sphere of $E_n$ containing $x$. Since the complement of $M$ in $E_n$ is of dimension at most $n - 2$, $M \cdot S$ is connected.‡ Likewise, it is locally connected. Being a $G_\delta$-subset of $E_n$ (it will be recalled that $M$ is a $G_\delta$-subset) it is, by a theorem of Alexandroff, in itself a complete space, and therefore arcwise connected. The sum of the infinite set of arcs $x_ix_j$ for every pair of *consecutive points* of $\cdots x_{-n}, \cdots, x_n, \cdots$, where $x_ix_j$ is an arc of $S_i \cdot M$ if $i$ is positive and of $S_{i+1}$ if $i$ is negative, plus the points $a$ and $b$, is a continuous curve containing $a$ and $b$, and has a subarc $ab$ of $M$ which has only the points $a$ and $b$ in common with $L'' - ab$ (of $L''$). It is clear that by ordering the intervals of $L$ complementary to $B$, and applying the above argument for a sequence of positive numbers $e$ converging to zero to the successively resulting rays we arrive

† G. T. Whyburn, "Concerning Continuous Curves Without Local Separating Point," *American Journal of Mathematics*, Vol. 53 (1931), p. 163.

‡ Urysohn, "Sur les multiplicités Cantoriennes," *Fundamenta Mathematicae*, Vol. 8 (1926), p. 310, Lemma. The restriction to closed sets is not essential.

at a ray $L$ contained in $M$, containing $B$, and with the endpoint $x$ of $L''$. Since $L \subset M$ and $M \cdot C \subset G'' + B \subset G + B$, $L \cdot C \subset G + B$. Since $B$ is closed, $G + B$ is zero-dimensional.† Since $L \cdot C = B^0$ is closed and zero-dimensional, it is totally disconnected. Since $B^0 \subset M \cdot C$, $B^0 \subset C - K$ and does not separate (locally separate) $C$. With this the theorem is proved.

If we suppose, as a very special case, that $B$ consists of two points $x$ and $y$ which do not belong to $C$, then we have shown that there is an arc $xy$ of $E_n$ whose product with $C$ is totally disconnected and does not even locally separate $C$. This generalises a result of R. L. Moore.‡ If we suppose that $C$ is any $F\sigma$-subset of $E_n$ of dimension at most $n-2$, and $B$ is any closed and totally disconnected subset of $C$, the ray $L$ has only $B$ in common with $C$. This generalizes a theorem due to Zarankiewicz on the accessibility from euclidean 3-space of the point of a one-dimensional continu.§ In particular if $a$ and $b$ are two points of $C$, and $ab$ is the straight line segment in $E_n$ there exists, by the argument which we gave on the subarc $ab$ of $L''$ (in the preceding theorem) an arc $ab$ which has only $a$ and $b$ in common with $C$ and whose diameter is the distance from $a$ to $b$. We shall make an application of this remark to the Moore-Kline Theorem.¶

If $M$ is a Moore-Kline set, bounded, in $E_n$: i. e. a closed set of points whose maximal connected subsets are points or arcs, and if $x$ in any point of an arc $X$ of $M$ but not an endpoint of $X$, then $x$ is not a limit point of $M - X$, then $M$ is a one-dimensional $F\sigma$-subset of $E_n$. For it is the sum of countable set of arcs and a closed totally disconnected point set consisting of endpoints of arcs of $M$ and point-components. There is in $E_n$ an acyclic continuous curve which contains $M$.‖ There is, by Waszewski-Gehman Theorem an acyclic continuous curve in the plane homeomorphic with the given acyclic continuous curve. By this there is a plane set $M''$ homeomorphic with $M$. By the Moore-Kline Theorem there is an arc $L''$ which contains $M''$. This imposes an ordering upon the points of $M''$, and therefore, by the correspondence, upon the points of $M$. If $a''_i b''_i$ is an arc of $L''$ whose

† K. Menger, *Dimensionstheorie*, p. 114: $B$ is at once a $G\delta$ and $F_\sigma$-set.

‡ R. L. Moore, "Concerning Paths That Do Not Separate a Given Continuous Curve," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), p. 750, Theorem 10.

§ C. Zarankiewicz, "Sur les points de division dans les ensembles connexes," *Fundamenta Mathematicae*, Vol. 9 (1927), p. 166 (3).

¶ R. L. Moore and J. R. Kline, "On the Most General Closed (Plane) Point Set Through Which it is Possible to Pass a Simple Continuous Arc," *Annals of Mathematics*, Vol. 20 (1919).

‖ L. Zippin, "On Continuous Curves and the Jordan Curve Theorem," *American Journal of Mathematics*, Vol. 52 (1930), p. 332, Theorem 2.

endpoints only are points of $M''$, we shall regard the corresponding point pair $a_i$ and $b_i$ of $M$ as associated. By the homeomorphism and the fact that $M$ and $M''$ are closed and bounded, it is seen that the distance in $E_n$ between associated pairs of points $a_n$ and $b_n$ approaches zero with increasing $n$: we suppose the denumerable set of intervals complementary to $M''$ in $L''$ arbitrarily ordered. Then $M$ plus the countable set of straight line intervals $a_nb_n$ is a one-dimensional $F_\sigma$-subset $M^0$ of $E_n$. Replacing the intervals $a_nb_n$ by independent arcs of $E_n \cdot \{M^* - (a_n + b_n)\}$ of the same diameter, as indicated above, we have an arc $L$ which contains $M$ and preserves the order imposed upon $M$ by the homeomorphism with $M''$. Except for the detail of "order" however, this is a known result.†

To obtain a different version of Cleveland's theorem,‡ and because it may be of interest in itself, we shall turn to a theorem on generalized continuous curves. We shall show, first, that if $B$ is a self-compact totally disconnected subset of a generalized continuous curve (i. e. a complete separable metric space, connected and connected im kleinen) and $B$ does not even locally separate $C$, then there exists in $C$ an acyclic continuous curve $T$ the set of whose endpoints is $B$. From the self-compactness of $B$ it readily follows that there exists in $C$ a finite set $U_1, U_2, \cdots, U_n$ of open connected sets of diameter at most 1, such that $\bar{U}_i \cdot \bar{U}_j = 0$, $U_i \cdot B \equiv B_i = \bar{B}_i$ (not vacuous), and $U_i - B_i$ is connected: therefore as we have seen in previous arguments, arcwise connected. If $k_i$, $1 \leq i \leq n$, is a set of points, $k_i \subset U_i - B_i$, there is in $C - B$ a tree $T_1$ irreducible about the set $k_i$. There exists in $U_i$, $1 \leq i \leq n$, a finite set of open connected subsets $U_{ij}$, $1 \leq j \leq n_i$, of diameter less than $\frac{1}{2}$, $\bar{U}_{ij} \cdot T_1 = 0$, $\bar{U}_{ij} \cdot \bar{U}_{ik} = 0$, $U_{ij} \cdot B = U_{ij} \cdot B_i \equiv B_{ij} = \bar{B}_{ij}$, and $U_{ij} - B_{ij}$ is arcwise connected. Let $k_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n_i$, be a finite set of points, $k_{ij} \subset U_{ij} - B_{ij}$. There is a tree (acyclic continuous curve) $T_2$ irreducible about the set $k_{ij}$, and $T_1 \supset T_2 \cdot (C - \Sigma v_i)$; the point set $T_1 \cdot T_2$ is itself a tree: it suffices, for each point $k_{ij}$ (these being arbitrarily ordered) to consider any arc $y_{i1}k_i$ in $U_i - B_i$, and the subarc on this from $k_{i1}$ to the first point on $T_1$; for succeeding points $k_{ij}$ the subarc from $k_{ij}$ to the first point on $T_1$ plus the finite set of arcs already added.§ We define inductively a sequence of trees, $T_n$; the point set compared of points common

† E. W. Miller, *Bulletin of the American Mathematical Society*, Abstract, Vol. 36 (1930), p. 361, 265.

‡ Announced by the author, some time ago, as a theorem for continuous curves.

§ This and other details of the argument parallel closely the proof which has been given of a more restricted theorem in L. Zippin, "A Study of Continuous Curves and Their Relation to the Janiszewski-Mullikin Theorem," *Transactions of the American Mathematical Society*, Vol. 31 (1930), p. 745, Theorem 1.

to all but a finite number of these being itself a locally compact acyclic continuous curve $T$, $\bar{T} - T = B$, $\bar{T}$ is an acyclic continuous curve (compact) the set of whose endpoints is $B$.‡

It may be remarked that in order that *every* self-compact, totally disconnected subset $B$ of a generalized continuous curve $C$ be the set of endpoints of an acyclic continuous subcurve of $C$, it is *necessary* (by the argument above, it is also sufficient) that no self-compact totally disconnected subset $H$ of $C$ even locally separate $C$. Otherwise there is an open connected subset $U$ of $C$, $U \cdot H = H'' \subset \bar{H}'' \subset U$, and $U - H''$ is not connected. There is a subset $H^0$ of $H''$ which separates $U$ and a point $h$ of $H^0$ which is a sequential limit point for two distinct sequences $(x_i)$ and $(y_i)$ of $U - H^0$, such that there is no arc $x_i y_j$ in $U - H^0$.§ But then it will be seen that no acyclic continuous curve of $C$ can have the point set $H^0 + (x_i) + (y_i)$ as the set of its endpoints: for some $j$ and $k$ the corresponding arc $x_j y_k$ must have at least one point of $H^0$ as inner point. Then we have the

THEOREM. *A necessary and sufficient condition that every self-compact totally disconnected subset of a generalized continuous curve $C$ be the set of endpoints of an acyclic continuous curve of $C$ is that no self-compact totally disconnected subset of $C$ even locally separate $C$.*

We return to a consideration of Cleveland's theorem and prove

THEOREM 2. *If $C$ is a continuous curve in $E_n$, $n > 2$, and contains no domain, and $B$ is a closed and totally disconnected subset of $E_n$ such that $B'' = B \cdot C$ does not separate (locally separate) $C$, then there exists an acyclic continuous curve $T$ in $E_n$ such that* 1) *$B$ is the set of endpoints of $T$,* 2) *$T \cdot C = B^0$ is totally disconnected,* 3) *$B^0$ does not separate (locally separate) $C$,* 4) *$B^0 - B \subset \bar{C}^{n-1}$.*

Consider the point set $M$ defined in the first theorem: it will be recalled that $M$ is a generalized continuous curve containing $B$ whose product with $C$ is a zero-dimensional set of points $G''$ such that no subset of $G'' + B''$ separates (locally separates) $C$. The complement of $M$ in $E_n$ is an $F_\sigma$-subset $K''$ (say) of dimension at most $n - 2$. Since $K'' + V$ is at most $(n-2)$-dimensional, where $V$ is any closed and totally disconnected subset of $M$, it is readily seen that $V$ does not even locally separate $M$. If, now, $B$ is

‡ The details are suppressed. The thought of the proof will be found, for example, in H. M. Gehman, "Concerning Acyclic Continuous Curves," *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 567, Theorem 6; also, reference above.

§ Compare the *preliminary* paragraph.

bounded in $E_n$ it is self-compact and we may apply the immediately preceding theorem to determine an acyclic continuous curve $T$ with the properties of the theorem (these follow as in Theorem 1). If $B$ is unbounded, we invert $E_n$ with respect to a center of inversion $v$ in $M$. Then $B^*$, the image of $B$, plus $v$ is a self-compact set of points: $M^*$, the image of $M$, is a generalized continuous curve. This follows from the fact that the image in $E_n$ of the $F_\sigma$-set $K''$ is an $F_\sigma$-set $K^*$ because the image of a closed set while not necessarily closed (without the addition of the center of inversion $v$, which we are reserving to $M^*$) is always an $F_\sigma$. Then there is an acyclic continuous curve $T^*$ in $M^*$ the set of whose endpoints is $B^* + v$, and the image of this in $E_n$ is an unbounded acyclic continuous curve $T$ whose endpoints are the set $B$. Again, that $T$ has all the properties required of it by the theorem follows as in Theorem 1.

We remark, finally, that if $C''$ is a continuous curve in $E_n$ and contains domains in $E_n$, there exists a continuous curve $C$ in $C''$ which is nowhere dense in $E_n$ and contains every boundary point of $C''$, and such that $C'' - C$ is a countable set of open spheres $(S_n)$ of $E_n \cdot C''$ the set of whose diameters converges to zero, and $\bar{S}_i \cdot \bar{S}_j = 0$.† This permits us to extend Theorems 1 and 2 to the continuous curves $C''$ which contain domains in $E_n$ if we replace the condition that the set $B^0$ defined in those theorems ($C''$ replacing $C$) is totally disconnected by this: if $X$ is any arc of points of $B^0$ and $x$ is an inner point of $X$, then $x$ belongs to one of the spheres $S_n$, above. The statements of the resulting theorems will be readily anticipated.

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† Compare in J. H. Roberts, "Concerning Non-Dense Plane Continua," *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 24 (2).

# TRITANGENT CIRCLES OF THE RATIONAL BICUBIC.

By Frank Morley and W. K. Morrill.

1. *Introduction.* The general sextic curve in a projective space is the intersection of a quadric surface and a cubic surface. With reference to the quadric surface it is a 3 to 3, or bicubic, correspondence along two generators. We take the quadric as a sphere and isolate a point $\infty$ of it. It is convenient to speak (inversively) of the sphere (with $\infty$ marked on it) as a plane. On it are conjugate coördinates $x$ and $\bar{x}$ and the curve is given by a self-conjugate, 3 to 3, relation between $x$ and $\bar{x}$. We call it a bicubic curve,—it is also, projectively, called a tricircular sextic. The tritangent planes of the projective curve are, on the plane, tritangent circles or contact circles.

In connection with a memoir on the tritangent planes by F. P. White, *Proceedings of the London Mathematical Society,* Vol. 30 (1930), p. 347; it occurred to J. H. Grace, *Journal of the London Mathematical Society,* Vol. 5 (1930), p. 121, and to Frank Morley, *American Mathematical Monthly,* Problem No. 3444, Vol. 37 (September 1930), that sets of four tritangent circles must have a common tangent circle. We discuss this, an extension of Hart's theorem,* for the case of the rational curve.

2. *Abel's theorem for sections by a circle.* The equation of a rational bicubic with four double points is given by,

$$x = \frac{\alpha_0 + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3}{\beta_0 + \beta_1\rho + \beta_2\rho^2 + \beta_3\rho^3}, \tag{1}$$

where $x$, $\alpha_i$, $\beta_i$ are complex quantities and $\rho$ is a real. If we substitute for $x$ and $\bar{x}$ from (1) and its conjugate in the equation $(x-c)(\bar{x}-\bar{c}) = r^2$ of a circle we get a sextic $f(\rho)$ in $\rho$, showing that a circle has with the curve six common points, either actual intersections or common image pairs. If we call $a_i$, $b_i$ $(i = 1, 2, 3, 4)$ the values of the parameter at the four double points, the six parameters of the points on a circle obey the relations,

$$\frac{(a_i-\rho_1)(a_i-\rho_2)(a_i-\rho_3)(a_i-\rho_4)(a_i-\rho_5)(a_i-\rho_6)}{(b_i-\rho_1)(b_i-\rho_2)(b_i-\rho_3)(b_i-\rho_4)(b_i-\rho_5)(b_i-\rho_6)} = k_i, \tag{2}$$

$$(i = 1, 2, 3, 4).$$

† Clebsch, *Vorlesungen über Geometrie der Ebene,* p. 893.

* Coolidge, *Circle and Sphere,* p. 43.

To determine $k_1$, we suppose the circle to pass through the three double points $(a_2, b_2)$, $(a_3, b_3)$ and $(a_4, b_4)$. Then we have,

$$k_1 = \frac{(a_1 - a_2)(a_1 - b_2)(a_1 - a_3)(a_1 - b_3)(a_1 - a_4)(a_1 - b_4)}{(b_1 - a_2)(b_1 - b_2)(b_1 - a_3)(b_1 - b_3)(b_1 - a_4)(b_1 - b_4)}.$$

Introducing $a_1 - b_1$ in the numerator and denominator and writing,

$$\phi(x) = (x - a_1)(x - b_1) \cdots (x - a_4)(x - b_4),$$

we have $k_1 = -\phi'(a_1)/\phi'(b_1)$, where $\phi'(x)$ represents the derivative of $\phi$ with respect to $x$. In like manner we evaluate $k_2$, $k_3$, and $k_4$. We can rewrite the equations (2) in the following way:

$$\frac{(a_1 - \rho_1) \cdots (a_1 - \rho_6)}{\phi'(a_1)} + \frac{(b_1 - \rho_1) \cdots (b_1 - \rho_6)}{\phi'(b_1)} = 0,$$

$$\frac{(a_2 - \rho_1) \cdots (a_2 - \rho_6)}{\phi'(a_2)} + \frac{(b_2 - \rho_1) \cdots (b_2 - \rho_6)}{\phi'(b_2)} = 0,$$

$$\frac{(a_3 - \rho_1) \cdots (a_3 - \rho_6)}{\phi'(a_3)} + \frac{(b_3 - \rho_1) \cdots (b_3 - \rho_6)}{\phi'(b_3)} = 0,$$

$$\frac{(a_4 - \rho_1) \cdots (a_4 - \rho_6)}{\phi'(a_4)} + \frac{(b_4 - \rho_1) \cdots (b_4 - \rho_6)}{\phi'(b_4)} = 0.$$

From the theory of partial fractions the sum of the above expressions is identically zero and therefore the equations (2) amount to three equations.

3. *The tritangent circles.* If we consider the circle to be tritangent to the bicubic, the values of the parameters of the points of intersection become equal by pairs. Suppose $\rho_1 = \rho_4$, $\rho_2 = \rho_5$, $\rho_3 = \rho_6$, then (2) becomes,

$$\frac{(a_i - \rho_1)^2 (a_i - \rho_2)^2 (a_i - \rho_3)^2}{(b_i - \rho_1)^2 (b_i - \rho_2)^2 (b_i - \rho_3)^2} = k_i$$

and we have

$$(3) \quad \begin{cases} \dfrac{(a_1 - \rho_1)(a_1 - \rho_2)(a_1 - \rho_3)}{(b_1 - \rho_1)(b_1 - \rho_2)(b_1 - \rho_3)} = (k_1)^{\frac{1}{2}}, \\ \dfrac{(a_2 - \rho_1)(a_2 - \rho_2)(a_2 - \rho_3)}{(b_2 - \rho_1)(b_2 - \rho_2)(b_2 - \rho_3)} = (k_2)^{\frac{1}{2}}, \\ \dfrac{(a_3 - \rho_1)(a_3 - \rho_2)(a_3 - \rho_3)}{(b_3 - \rho_1)(b_3 - \rho_2)(b_3 - \rho_3)} = (k_3)^{\frac{1}{2}}, \\ \dfrac{(a_4 - \rho_1)(a_4 - \rho_2)(a_4 - \rho_3)}{(b_4 - \rho_1)(b_4 - \rho_2)(b_4 - \rho_3)} = (k_4)^{\frac{1}{2}}. \end{cases}$$

Now we have,

$$k_1k_2k_3k_4 = \prod^{6}(a_1 - a_2) / \prod^{6}(b_1 - b_2)^2.$$

and therefore,

$$(k_1)^{\frac{1}{2}}(k_2)^{\frac{1}{2}}(k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}}$$
$$= \pm \frac{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_2 - a_3)(a_2 - a_4)(a_3 - a_4)}{(b_1 - b_2)(b_1 - b_3)(b_1 - b_4)(b_2 - b_3)(b_2 - b_4)(b_3 - b_4)};$$

or writing $| a_1 a_2 a_3 a_4 |$ for the determinant

$$\begin{vmatrix} a_1^3 & a_1^2 & a_1 & 1 \\ a_2^3 & a_2^2 & a_2 & 1 \\ a_3^3 & a_3^2 & a_3 & 1 \\ a_4^3 & a_4^2 & a_4 & 1 \end{vmatrix}$$

we have

$$(4) \qquad (k_1)^{\frac{1}{2}}(k_2)^{\frac{1}{2}}(k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}} = \pm | a_1 a_2 a_3 a_4 | / | b_1 b_2 b_3 b_4 |;$$

and the question is to determine the sign. If we eliminate from (3) the symmetric functions of $\rho_i$ we have a determinant whose first row is,

$$a_1^3 - (k_1)^{\frac{1}{2}}b_1^3, \qquad a_1^2 - (k_1)^{\frac{1}{2}}b_1^2, \qquad a_1 - (k_1)^{\frac{1}{2}}b_1, \qquad 1 - (k_1)^{\frac{1}{2}}.$$

Expanded, this consists of pairs of determinants such as:

$$| a_1 a_2 a_3 a_4 | + (k_1)^{\frac{1}{2}}(k_2)^{\frac{1}{2}}(k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}} | b_1 b_2 b_3 b_4 |$$
$$- (k_1)^{\frac{1}{2}} | b_1 a_2 a_3 a_4 | - (k_2)^{\frac{1}{2}}(k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}} | a_1 b_2 b_3 b_4 |$$
$$+ (k_1)^{\frac{1}{2}}(k_2)^{\frac{1}{2}} | b_1 b_2 a_3 a_4 | + (k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}} | a_1 a_2 b_3 b_4 |$$

and each pair vanishes when

$$(4') \qquad (k_1)^{\frac{1}{2}}(k_2)^{\frac{1}{2}}(k_3)^{\frac{1}{2}}(k_4)^{\frac{1}{2}} = - | a_1 a_2 a_3 a_4 | / | b_1 b_2 b_3 b_4 |.$$

For, in this notation,

$$k_1 = \frac{| a_1 a_2 a_3 a_4 |}{| b_1 b_2 b_3 b_4 |} \cdot \frac{| a_1 b_2 b_3 b_4 |}{| b_1 a_2 a_3 a_4 |},$$

so that the second pair row vanishes, and

$$k_1k_2 = \frac{| a_1 a_2 a_3 a_4 |}{| b_1 b_2 b_3 b_4 |} \cdot \frac{| a_1 a_2 b_3 b_4 |}{| b_1 b_2 a_3 a_4 |},$$

so that the third pair row vanishes. We must then in (4) pick the negative sign. When we select $(k_1)^{\frac{1}{2}}$, $(k_2)^{\frac{1}{2}}$, and $(k_3)^{\frac{1}{2}}$, $(k_4)^{\frac{1}{2}}$ is now uniquely determined. Three of the equations (3) determine the fourth; and, since we can choose $(k_1)^{\frac{1}{2}}$, $(k_2)^{\frac{1}{2}}$, $(k_3)^{\frac{1}{2}}$ in 8 ways, there are 8 tritangent circles.

If we denote one of these by $+, +, +, +$ instead of by $(k_1)^{\frac{1}{2}}$, $(k_2)^{\frac{1}{2}}$, $(k_3)^{\frac{1}{2}}$, $(k_4)^{\frac{1}{2}}$, then there are, by equation (4), six like $+, +, -, -$, and the eighth is $-, -, -, -$.

4. *The biquadratics.* Consider now a biquadratic curve given by a self-conjugate equation of the second degree in $x$ and in $\bar{x}$. Substituting for $x$ and $\bar{x}$ from (1) we have an equation of degree 12 in $\rho$; so that the bicubic and the biquadratic have 12 common points.

As before, Abel's theorem takes the form

$$\frac{(a_i - \rho_1) \cdots (a_i - \rho_{12})}{(b_i - \rho_1) \cdots (b_i - \rho_{12})} = c_i.$$

To determine $c_i$ we consider the circle on the other double points, taken twice. We obtain thus $c_i = k_i^2$.

Thus the four relations on the common points are

$$\frac{(a_i - \rho_1) \cdots (a_i - \rho_{12})}{(b_i - \rho_1) \cdots (b_i - \rho_{12})} = k_i^2. \tag{5}$$

These are independent relations.

If now we write the conditions for four contact circles

$$\frac{(a_i - \rho_1)(a_i - \rho_2)(a_i - \rho_3)}{(b_i - \rho_1)(b_i - \rho_2)(b_i - \rho_3)} = \pm (k_i)^{\frac{1}{2}},$$

$$\frac{(a_i - \rho_4)(a_i - \rho_5)(a_i - \rho_6)}{(b_i - \rho_4)(b_i - \rho_5)(b_i - \rho_6)} = \pm (k_i)^{\frac{1}{2}},$$

$$\frac{(a_i - \rho_7)(a_i - \rho_8)(a_i - \rho_9)}{(b_i - \rho_7)(b_i - \rho_8)(b_i - \rho_9)} = \pm (k_i)^{\frac{1}{2}},$$

$$\frac{(a_i - \rho_{10})(a_i - \rho_{11})(a_i - \rho_{12})}{(b_i - \rho_{10})(b_i - \rho_{11})(b_i - \rho_{12})} = \pm (k_i)^{\frac{1}{2}},$$

the relations (5) are satisfied, if we take an even number of negative signs.

Taking four such circles, for example

| | | | | |
|---|---|---|---|---|
| 1) | + | + | + | + |
| 2) | − | + | + | − |
| 3) | + | − | + | − |
| 4) | − | − | + | + |

their 12 points of contact are on a biquadratic. Four such circles are called a set.

Any three circles have 9 points of contact on a biquadratic. For the above rule of signs determines the fourth circle. We have an even number of negative signs in each row by equation (4), so that if we have an even number in three columns we must have an even number in the fourth.

We have now 8 circles, and 56 ways of selecting three. With each three is a fourth. Hence there are 14 sets of four, that is 14 biquadratics.

5. *The extension of Hart's theorem.* Let us denote the form in $x$ and $\bar{x}$ which gives the bicubic by $C$, that for one of the biquadratics by $B$, and that for the set of four circles by $A_1$, $A_2$, $A_3$, $A_4$. Then since $B$ is on the points of contact of $A_i$ and $C$, there is an identity

$$A_1A_2A_3A_4 - B^2 = CA, \tag{6}$$

where $A$ is a bilinear form. Hence $A_1$ touches $CA$ where it meets $B$. It already touches $C$ three times; it therefore touches $A$ at the fourth intersection of $A_1$ and $B$. This is also true for $A_2$, $A_3$, and $A_4$. Therefore the circles of a set are touched by a circle. Since there are fourteen distinct biquadratics, there are fourteen such circles.

6. *The general rational curve.* The argument of this paper applies to any rational curve on a sphere. We have now

$$x = (\alpha\rho)^n/(\beta\rho)^n$$

with $(n-1)^2$ double points. Let the parameters at these be $a_i$, $b_i$. Omitting one double point, there is on the $n(n-2)$ others a definite curve

$$f^{n-2} \equiv f(x^{n-2}, \bar{x}^{n-2}) = 0.$$

This does not meet the given curve elsewhere. For the general $f^{n-2}$ Abel's theorem is

$$\prod_i{}^{2n(n-2)} \equiv \frac{(a_i-\rho_1)\cdots(a_i-\rho_{2n(n-2)})}{(b_i-\rho_1)\cdots(b_i-\rho_{2n(n-2)})} = k_i, \qquad [i = 1\cdots(n-1)^2].$$

As in § 2 these $(n-1)^2$ equations amount only to $n(n-2)$.

For a contact curve we have

$$\prod_i{}^{n(n-2)} \equiv \frac{(a_i-\rho_1)\cdots(a_i-\rho_{n(n-2)})}{(b_i-\rho_1)\cdots(b_i-\rho_{n(n-2)})} = (k_i)^{\frac{1}{2}}.$$

By the determinant argument of § 3, the product of these selected square roots is definitive. We can only change the sign of an even number, therefore the number of contact curves is, writing $\mu$ for $(n-1)^2$,

$$1 + \binom{\mu}{2} + \binom{\mu}{4} + \cdots,$$

that is $2^{\mu-1}$ or $2^{n(n-2)}$. They fall again into sets of 4 which lie on a $f^{2(n-2)}$. And the identity of § 5 now becomes

$$f_1{}^{n-2}f_2{}^{n-2}f_3{}^{n-2}f_4{}^{n-2} - [f^{2(n-2)}]^2 = f^n f^{3n-8}.$$

# INTEGRAL EQUATIONS AND THE COOLING PROBLEM FOR SEVERAL MEDIA.

By W. M. Rust, Jr.

## PART I. Solution of a Generalized Abel Integral Equation.

In a simple problem concerning the cooling of castings, the determination of the temperature may be reduced to the solution of a certain type of generalized Abel integral equation. The purpose of this first part is to discuss a method of solving this equation.

The equation can be put in the form

$$(1.1)\quad \int_0^t \left\{ \frac{1+K(t,t')}{(t-t')^{\frac{1}{2}}} \right\} u(t')\,dt' = f(t) \text{ with } K(t,t') \text{ and } f(t) \text{ known.}$$

In the actual problem, the function $K(t,t')$ vanishes to a high order for $t=t'$. For the method used here it is sufficient that $K(t,t')$ satisfy the conditions;

(A) $K(t,t')/(t-t')$ is bounded and is absolutely continuous in $t$, uniformly for all $t' \leq t$, and is summable in each of the variables. This implies the vanishing to at least the first order of $K(t,t')$ for $t=t'$.

(B) $\partial K(t,t')/\partial t'$ exists, is bounded and is absolutely continuous in $t$, uniformly for all $t' \leq t$, and is summable in each of the variables.*

The function $f(t)$ is assumed bounded and absolutely continuous, it is *not* assumed that $\underset{t=0}{\text{Limit}}\, f(t) = 0$, as is usually a condition for solution.

We shall assume that the equation (1.1) has a summable solution and

* The form (1.1) was used because this is the form needed in the sequel, however no essential change is made if we write the equation in the form

$$\int_0^t [G(t,t')/(t-t')^{\lambda}]\, u(t')\,dt' = f(t).$$

Where $G(t,t')$ satisfies the following conditions

(1) $G(t,t')$ is bounded and is absolutely continuous in $t$, uniformly for all $t' \leq t$, and is summable in each of the variables.

(2) For $t'$ near enough to $t$, we have $G(t,t') = g(t) + h(t)(t-t')$ plus higher powers of $(t-t')$, and $|g(t)| \geq m > 0$, and $g(t)$ and $h(t)$ are bounded and absolutely continuous.

(3) $\partial G(t,t')/\partial t'$ exists, is bounded and is absolutely continuous in $t$, uniformly for all $t' \leq t$, and is summable in each of the variables, $\lambda$ is a constant between zero and one, $f(t)$ is bounded and absolutely continuous.

shall solve for $\int_0^{t''} u(t)\,dt$, rather than for $u(t)$ itself. The method of solution is to multiply through by $(t''-t)^{-\frac{1}{2}}$ and integrate from $t=0$ to $t=t''$, as in the usual treatment. If (1.1) is satisfied by $u(t)$, this gives

$$(1.2)\qquad \int_0^{t''} \frac{dt}{(t''-t)^{\frac{1}{2}}} \int_0^{t} \left\{ \frac{1+K(t,t')}{(t-t')^{\frac{1}{2}}} \right\} u(t)\,dt' \equiv \int_0^{t''} \frac{f(t)\,dt}{(t''-t)^{\frac{1}{2}}}.$$

In the left hand member we change the order of integration, which is allowable since $K(t,t')$ is bounded. The result is

$$\int_0^{t''} \frac{dt}{(t''-t)^{\frac{1}{2}}} \int_0^{t} \left\{ \frac{1+K(t,t')}{(t-t')^{\frac{1}{2}}} \right\} u(t')\,dt'$$
$$\equiv \pi \int_0^{t''} u(t')\,dt' + \int_0^{t''} u(t')\,dt' \int_{t'}^{t''} \frac{K(t,t')\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}}.$$

At this point we vary from the usual treatment by integrating the last term by parts. This gives

$$\left[ \left\{ \int_0^{t'} u(t)\,dt \right\} \left\{ \int_{t'}^{t''} \frac{K(t,t')\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right\} \right]_{t'=0}^{t'=t''}$$
$$- \int_0^{t''} \left\{ \int_0^{t'} u(t)\,dt \right\} \frac{\partial}{\partial t'} \left( \int_{t'}^{t''} \frac{K(t,t')\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right) dt'.$$

The term outside the sign of integration vanishes since the factor $\int_0^{t'} u(t)\,dt$ vanishes for $t'=0$ and the factor $\int_{t'}^{t''} \frac{K(t,t')\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}}$ vanishes for $t'=t''$. This follows from condition (A), for if $K(t,t')$ vanishes to the order $\alpha$ for $t=t'$, this integral is less than

$$\int_{t'}^{t''} \frac{M\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}-\alpha}} \equiv M(t''-t')^{\alpha} B(\tfrac{1}{2}, \tfrac{1}{2}+\alpha)$$

and vanishes to the order $\alpha$ for $t'=t''$.

Formal differentiation gives

$$(1.3)\qquad \frac{\partial}{\partial t'} \left( \int_{t'}^{t''} \frac{K(t,t')\,dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right) \equiv \left[ \frac{-K(t,t')}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} \right]_{t=t''}$$
$$+ \int_{t'}^{t''} \left\{ \frac{(\partial/\partial t')K(t,t')}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}} + \frac{K(t,t')}{2(t''-t)^{\frac{1}{2}}(t-t')^{3/2}} \right\} dt.$$

The first term is zero by condition (A). The second term by condition (B) is bounded. The third term is bounded, for by condition (A) the function $K(t,t')$ vanishes to at least the first order for $t=t'$ and so the integral is less than

$$M\int_{t'}^{t''}(t''-t)^{-\frac{1}{2}}(t-t')^{-3/2+\alpha}dt$$

with $\alpha$ at least as great as one.

This establishes the fact that the derivative (1.3) is bounded and justifies the integration by parts.

We thus have as a necessary condition on solutions of (1.1),

$$(1.4)\quad \pi\int_0^{t''}u(t')dt' \equiv \int_0^{t''}\frac{f(t')dt'}{(t''-t')^{\frac{1}{2}}} + \int_0^{t''}\left\{\int_0^{t'}u(t)dt\right\}\frac{\partial}{\partial t'}\left(\int_{t'}^{t''}\frac{K(t,t')dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}}\right)dt'.$$

We wish to show that the equation

$$(1.5)\quad \pi V(t'') = \int_0^{t''}\frac{f(t')dt'}{(t''-t')^{\frac{1}{2}}} + \int_0^{t''}V(t')\frac{\partial}{\partial t'}\left(\int_{t'}^{t''}\frac{K(t,t')dt}{(t''-t)^{\frac{1}{2}}(t-t')^{\frac{1}{2}}}\right)dt'$$

has a unique absolutely continuous solution, whose derivative satisfies the equation (1.1) nearly everywhere.

Since equation (1.5) is a Volterra integral equation of the second sort with a bounded kernel, (1.3), it may be solved by a process of successive approximations, the process necessarily converging and the solution being summable and unique.*

Since the equation (1.4) is a necessary condition on summable solutions of (1.1), the uniqueness of the solution of (1.5) shows that the solutions of (1.1) can, at most, differ on a point set of zero measure.

To show that the solution of (1.5) is absolutely continuous, we need a lemma.

LEMMA I. *The function* $F(t)=\int_0^1 f(t,t')g(t')dt'$ *is an absolutely continuous function of* $t$ *if* $f(t,t')$ *is absolutely continuous in* $t$, *uniformly for all* $t'$ *and if* $g(t')$ *is a summable function.*

*Proof.* By the $\lambda$-variation of the function $f(t,t')$ we shall mean the function $\tau_f(\lambda)$ defined as the upper limit, for all sets of non-overlapping intervals $(a_i,b_i)$ such that $\Sigma\,|\,a_i-b_i\,|\leq\lambda$ and for all $t'$, of the numbers

$$\Sigma\,|\,f(a_i,t')-f(b_i,t')\,|\,.$$

* Volterra, *Leçons sur les Équations Intégrales*, p. 40 ff.

To say that $f(t, t')$ is absolutely continuous in $t$, uniformly for all $t'$, means that the $\lambda$-variation approaches zero with $\lambda$.

The $\lambda$-variation of $F(t)$, defined in an analogous way, is less than or equal to

$$\tau_f(\lambda) \int_0^1 g(t')dt'.$$

This approaches zero with $\lambda$, so that $F(t)$ is absolutely continuous. This establishes the Lemma.*

By means of the transformation $t' = t''y$, the first term on the right hand side of equation (1. 5) becomes

$$(t'')^{\frac{1}{2}} \int_0^1 f(t''y)\,(1-y)^{-\frac{1}{2}}\,dy$$

which satisfies the condition of the Lemma I and so is absolutely continuous.

By means of the transformation $t = t' + y(t'' - t')$, the expression (1. 3) becomes

$$\int_0^1 \left\{ \frac{K_2(t' + y\,[t'' - t'], t')}{(1-y)^{\frac{1}{2}}\, y^{\frac{1}{2}}} + \frac{K(t' + y\,[t'' - t'], t')}{2(1-y)^{\frac{1}{2}} y^{3/2}(t'' - t')} \right\} dt$$

where $K_2(t, t') = \partial K(t, t')/\partial t'$. By condition (B) and the Lemma I, the first term is absolutely continuous in $t''$. By condition (A) $K/(t - t') = K/y(t'' - t')$ is absolutely continuous in $t''$, and bounded so that by the Lemma I the second term is absolutely continuous in $t''$. In each case the absolute continuity is uniform in $t'$. Thus the expression (1. 3) is absolutely continuous in $t''$, uniformly in $t'$.

We need now the further Lemma,

LEMMA II. *The function* $G(t'') = \int_0^{t''} g(t')k(t'', t')dt'$ *is an absolutely continuous function of* $t''$, *if* $k(t'', t')$ *is absolutely continuous in* $t''$ *uniformly for all* $t'$ *and bounded, and if* $g(t')$ *is summable.*

*Proof.* The $\lambda$-variation, $\tau_G(\lambda)$, of $G(t'')$ is the upper limit for all sets of non-overlapping intervals $(a_i, b_i)$ such that $\Sigma \mid a_i - b_i \mid \leq \lambda$ of the numbers

$$\Sigma \mid \int_0^{b_i} g(t')k(b_i, t')dt' - \int_0^{a_i} g(t')k(a_i, t')dt' \mid$$

$$= \Sigma \mid \int_{a_i}^{b_i} g(t')k(b_i, t')dt' + \int_0^{a_i} g(t')\{k(b_i, t') - k(a_i, t')\}dt' \mid .$$

* The condition is a sufficient one, regardless of the manner in which $f(t, t')$ involves $t$, but in general it is not a necessary condition. If $f(t, t')$ is a function of $t$ alone, the condition is obviously necessary, but if $f(t, t')$ is a function of the product $tt'$, the condition may be modified to admit a set of points of discontinuity of the first sort, if this set is of zero measure.

Thus

$$\tau_G(\lambda) \leq \overline{\text{Limit}}\, \Sigma M \int_{a_i}^{b_i} |\, g(t')\,|\, dt' + \tau_k(\lambda) \int_0^{T''} |\, g(t')k\,|\, t',$$

$M$ is maximum absolute value of $k(t, t')$,

with $T''$ the greatest of the numbers $a_i$ or $b_i$. Since $g(t')$ is summable, each of these terms approaches zero with $\lambda$. This establishes the Lemma.

The second term in the right-hand member of (1.5) satisfies the conditions of Lemma II and so is absolutely continuous. Since each of the terms in the right-hand member of (1.5) is absolutely continuous, $V(t'')$ is absolutely continuous as was to be shown.

Since $V(t)$ is absolutely continuous, it has a derivative $V'(t)$ nearly everywhere and $\int_0^t V'(t')dt' = V(t)$. We shall now show that $u(t) = V'(t)$ satisfies equation (1.1) nearly everywhere. To do this we write

$$D(t) = \int_0^t \left\{ \frac{1 + K(t, t')}{(t - t')^{\frac{1}{2}}} V'(t') \right\} dt' - f(t). \tag{1.6}$$

Multiplying through by $(t'' - t)^{-\frac{1}{2}}$, integrating from $t = 0$ to $t = t''$, changing the order of integration and integrating by parts, as above, gives

$$\int_0^{t''} \frac{D(t)dt}{(t'' - t)^{\frac{1}{2}}} \equiv \pi \int_0^{t''} V'(t)dt - \int_0^{t''} \frac{f(t)dt}{(t'' - t)^{\frac{1}{2}}} \tag{1.7}$$

$$- \int_0^{t''} \left\{ \int_0^{t'} V'(t)dt \right\} \frac{\partial}{\partial t'} \left( \int_{t'}^{t''} \frac{K(t, t')dt}{(t'' - t)^{\frac{1}{2}}(t - t')^{\frac{1}{2}}} \right) dt' \equiv \pi V(t'')$$

$$- \int_0^{t''} \frac{f(t)dt}{(t'' - t)^{\frac{1}{2}}} - \int_0^{t''} V(t') \frac{\partial}{\partial t'} \left( \int_{t'}^{t''} \frac{K(t, t')dt}{(t'' - t)^{\frac{1}{2}}(t - t')^{\frac{1}{2}}} \right) dt'.$$

But since $V(t)$ satisfies equation (1.5), the right-hand side of (1.7) is zero. Hence, as Tonelli * shows, $D(t)$ is zero nearly everywhere, that is, the derivative of the solution of equation (1.5) satisfies (1.1) nearly everywhere.

We have the following theorem.

THEOREM. *Under the conditions specified with respect to $K(t, t')$ and $f(t)$ in (A) and (B), the equation (1.1) has one and only one summable solution $u(t)$, and this solution is given as the derivative of the solution of equation (1.5).*

Two summable functions differing only on a point set of zero measure are, of course, to be considered equivalent.

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* L. Tonelli, "Su un problema di Abel," *Mathematische Annalen*, Vol. 99 (1928), p. 187.

## PART II. A Problem in the Flow of Heat.

As an example of a physical problem that may be solved by such an equation, we consider the following problem in heat conduction, one which has both theoretical and intrinsic interest.

A quantity of one material is heated and placed between two masses of a different material, as a casting in its mold. The problem is: given the initial temperatures of the three regions and the temperatures at the two outer boundaries, find the temperature at any point at any time.

For convenience we consider very simple conditions. We take the three regions to be bounded by parallel infinite planes and take the thicknesses of the two outer regions to be equal. We take the initial temperatures in the outer regions to be constant and equal to each other. We take the initial temperature of the inner region to be constant, not necessarily the same as in the outer regions. We take the temperatures on the outer bounding surfaces to be equal and, at any instant, constant over the entire bounding plane. With these conditions the problem is symmetric about the central plane and at any instant the temperature is constant over any plane parallel to the central plane, thus only one spatial coördinate, the distance from the central plane, is involved. Some generalizations of this problem are considered in Part V.

The conductivity of the material in the inner region is $K_1$ and the conductivity of the material in the outer regions is $K_2$. The quantities $a^2$ and $b^2$ are positive constants, equal to the ratios of the conductivity to the product of the specific heat by the density, for the inner and outer regions respectively. Also, $x$ is the distance from the central plane and $t$ is the time after the initial time.

We take the bounding planes between the inner and outer regions to be at $x = m$ and $x = -m$ and the outer boundaries to be at $x = l$ and $x = -l$.

At interior points, $u_1(x, t)$, the temperature in the inner region, and $u_2(x, t)$, the temperature in the outer regions, satisfy the partial differential equations

$$(2.1)\quad \begin{cases} \partial^2 u_1(x,t)/\partial x^2 - a^2(\partial u_1(x,t)/\partial t) = 0 & -m < x < m \\ \partial^2 u_2(x,t)/\partial x^2 - b^2(\partial u_2(x,t)/\partial t) = 0 & -l < x < -m \text{ or } m < x < l \end{cases}$$

respectively.*

* For the derivation of the equations (2.1) see any standard work on the conduction of heat, for example Carslaw, *Conduction of Heat*, Chapter One, or Riemann-Weber, *Differentialgleichungen der Physik*, Vol. 2 (1912), p. 82.

If the temperature in the inner region is initially $u_1$, a constant, and in the outer regions $u_2$, a constant, we have

$$(2.2)\quad \begin{cases} \underset{t=0+}{\text{Limit}}\ u_1(x,t) = u_1 & \text{for} \quad -m < x < m \\ \underset{t=0+}{\text{Limit}}\ u_2(x,t) = u_2 & \text{for} \quad -l < x < -m \quad \text{or} \quad m < x < l. \end{cases}$$

At the outer boundaries the temperature is taken to be a known, bounded, continuous function of the time, say $f(t)$, with a bounded derivative, that is

$$(2.3)\qquad \underset{x=l-0}{\text{Limit}}\ u_2(x,t) = f(t) \quad \text{for} \quad t > 0.$$

At the separating boundaries we have two conditions, first, the temperature is continuous in $x$ across the boundaries, that is,

$$(2.4)\qquad \underset{x=m-0}{\text{Limit}}\ u_1(x,t) = \underset{x=m+0}{\text{Limit}}\ u_2(x,t) \quad \text{for} \quad t > 0,$$

and, second, the partial derivatives with respect to $x$ satisfy the equation †

$$(2.5)\quad \underset{x=m-0}{\text{Limit}}\ K_1 \partial u_1(x,t)/\partial x = \underset{x=m+0}{\text{Limit}}\ K_2 \partial u_2(x,t)/\partial x \quad \text{for} \quad t > 0.$$

Similar conditions are imposed at $x = -m$ and at $x = -l$, but due to the symmetry of the problem, these are equivalent to those given by the equations (2.3) to (2.5).

We establish the following uniqueness theorem.

UNIQUENESS THEOREM A. *There can not be more than one solution of the problem as given by equations* (2.1), *subject to the conditions* (2.2) *to* (2.5), *which is bounded everywhere* (*including* $t = 0$), *is continuous for* $t > 0$, *except at the boundaries, and has first derivatives with respect to each of the variables, which are continuous for* $t > 0$, *except at the boundaries—the derivative with respect to* $x$ *being bounded everywhere for* $t > 0$.

Suppose there were two such solutions. Call the difference between these solutions $V_1(x,t)$ and $V_2(x,t)$ in the inner and outer regions respectively. $V_1(x,t)$ and $V_2(x,t)$ satisfy the conditions of the theorem and take on the boundary and initial values zero.

Consider the integrals

$$(a)\qquad J_\epsilon(t) = (K_2 b^2/2) \int_{-l+\epsilon}^{-m-\epsilon} (V_2(x,t))^2\, dx + (K_1 a^2/2) \int_{-m+\epsilon}^{m-\epsilon} (V_1(x,t))^2\, dx + (K_2 b^2/2) \int_{m+\epsilon}^{l-\epsilon} (V_2(x,t))^2\, dx.$$

† For the derivation of equations (2.5) see, for example, Riemann-Weber, Vol. 2 (1912), p. 85, cited above.

For $t > 0$, $V_i(x,t)$ and $\partial V_i(x,t)/\partial t$ are continuous and bounded in the region of integration, so that we may differentiate under the integral sign. We have

$$\frac{dJ_\epsilon(t)}{dt} = K_2 b^2 \int_{-l+\epsilon}^{-m-\epsilon} V_2(x,t)\ \frac{\partial V_2}{\partial t}\ (x,t)\,dx$$

$$+ K_1 a^2 \int_{-m+\epsilon}^{m-\epsilon} V_1(x,t)\ \frac{\partial V_1}{\partial t}\ (x,t)\,dx + K_2 b^2 \int_{m+\epsilon}^{l-\epsilon} V_2(x,t)\ \frac{\partial V_2}{\partial t}\ (x,t)\,dx$$

and

$$\text{(b)} \qquad \int_{t_1}^{t_2} \frac{dJ_\epsilon(t)}{dt}\ dt = J_\epsilon(t_2) - J_\epsilon(t_1), \qquad t_1, t_2, > 0$$

applying (2.1), we have

$$\frac{dJ_\epsilon(t)}{dt} = K_2 \int_{-l+\epsilon}^{-m-\epsilon} V_2(x,t)\ \frac{\partial^2 V_2}{\partial x^2}\ (x,t)\,dx$$

$$+ K_1 \int_{-m-\epsilon}^{m-\epsilon} V_1(x,t)\ \frac{\partial^2 V_1}{\partial x^2}\ (x,t)\,dx + K_2 \int_{m+\epsilon}^{l-\epsilon} V_2(x,t)\ \frac{\partial^2 V_2}{\partial x^2}\ (x,t)\,dx.$$

In the regions of integration each $\partial^2 V_i/\partial x^2$ is continuous and bounded, so that we may integrate by parts.

$$\text{(c)} \quad dJ_\epsilon/dt = K_2 V_2(x,t)\,(\partial V_2/\partial x)\,(x,t)\Big]_{x=-l+\epsilon}^{x=-m-\epsilon}$$

$$+ K_1 V_1(x,t)\,(\partial V_1/\partial x)\,(x,t)\Big]_{x=-m-\epsilon}^{x=+m-\epsilon} + K_2 V_2(x,t)\,(\partial V_2/\partial x)\,(x,t)\Big]_{x=m+\epsilon}^{x=l-\epsilon}$$

$$- K_2 \int_{-l+\epsilon}^{-m-\epsilon} ((\partial V_2/\partial x)\,(x,t))^2\,dx - K_1 \int_{-m+\epsilon}^{+m-\epsilon} ((\partial V_1/\partial x)\,(x,t))^2\,dx$$

$$- K_2 \int_{m+\epsilon}^{l-\epsilon} ((\partial V_2/\partial x)\,(x,t))^2\,dx.$$

We have then from (b) and (c),

$$\text{(d)} \quad \operatorname*{Limit}_{\epsilon=0}\ [J_\epsilon(t_2) - J_\epsilon(t_1)] = \operatorname*{Limit}_{\epsilon=0} \int_{t_1}^{t_2} (dJ_\epsilon/dt)\,dt$$

$$= \operatorname*{Limit}_{\epsilon=0} \int_{t_1}^{t_2} \Big\{ K_2 V_2(x,t)\,(\partial V_2/\partial x)\,(x,t)\Big]_{x=-l+\epsilon}^{x=-m-\epsilon}$$

$$+ K_1 V_1(x,t)\,(\partial V_1/\partial x)\,(x,t)\Big]_{x=-m+\epsilon}^{x=+m-\epsilon} + K_2 V_2(x,t)\,(\partial V_2/\partial x)\,(x,t)\Big]_{x=m+\epsilon}^{x=l-\epsilon}$$

$$- K_2 \int_{-l+\epsilon}^{-m-\epsilon} ((\partial V_2/\partial x)\,(x,t))^2\,dx - K_1 \int_{-m+\epsilon}^{+m-\epsilon} (\partial V_1/\partial x)\,(x,t))^2\,dx$$

$$- K_2 \int_{m+\epsilon}^{l-\epsilon} ((\partial V_2/\partial x)\,(x,t))^2\,dx \Big\}\,dt.$$

Since each $V_i(x,t)$ is bounded

$$\text{(e)} \quad \lim_{\epsilon=0} J_\epsilon(t) = J(t) = K_2 b^2/2 \int_{-l}^{-m} (V_2(x,t))^2\, dx + K_1 a^2/2 \int_{-m}^{+m} (V_1(x,t))^2\, dx + K_2 b^2/2 \int_{m}^{l} (V_2(x,t))^2\, dx.$$

The left-hand member of (d) is thus

$$J(t_2) - J(t_1).$$

In the right-hand member of (d), we may pass to the limit under the sign of integration in the first three terms, since $V_i(x,t)$ and $\partial V_i/\partial x(x,t)$ are bounded.* That is

$$\text{(f)} \quad \lim_{x=x_0\pm 0} \int_{t_1}^{t_2} K_i V_i(x,t)\,(\partial V_i/\partial x)\,(x,t)\,dt = \int_{t_1}^{t_2} \lim_{x=x_0\pm 0} \{K_i V_i(x,t)\,(\partial V_i/\partial x)\,(x,t)\}\,dt$$

$x_0$ being the $x$ coördinate of any boundary. But by the boundary conditions the limit of the first three terms is zero.†

Since $K_1$ and $K_2$ are positive constants, the remaining three terms are negative, or zero, if $t_2 > t_1$, so that in the limit the right-hand member of (d) is negative or zero.

We have then

$$\text{(g)} \quad J(t_2) - J(t_1) \leq 0 \quad \text{if} \quad t_2 \geq t_1 > 0.$$

Since $[V_i(x,t)]^2$ is bounded and $\lim_{t=0} V_i(x,t) = 0$, nearly everywhere, we have

$$\lim_{t=0} J(t) = 0.$$

Applying this to (g) we have

$$\text{(h)} \quad J(t) \leq 0, \quad \text{if} \quad t > 0.$$

But in (e) the terms are all positive, since $K_1$ and $K_2$ are positive, so that

$$\text{(j)} \quad J(t) \geq 0, \quad \text{if} \quad t > 0.$$

We see (h) and (j) can only be satisfied simultaneously if

$$J(t) = 0.$$

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* See Lebesgue, *Leçons sur L'integration* (1928), p. 125.

† For infinite outer regions, we may replace (2.3) by the condition

$$\lim (\partial u/\partial x)\,(x,t) \equiv 0, \quad t > 0.$$

But from (e) this can only be the case if $V_i(x, t)$ is zero nearly everywhere and since $V_i(x, t)$ is continuous except at the boundaries for $t > 0$, it must be identically zero for $t > 0$ and the two solutions are identical for $t > 0$. *This establishes the theorem.*

We now establish a uniqueness theorem with less restrictive conditions.

UNIQUENESS THEOREM B. *There can not be more than one solution of the problem satisfying the conditions of Theorem A, except that the derivative with respect to $x$ is not assumed bounded, but summable in $t$ and satisfying the condition*

$$\operatorname*{Limit}_{x=x_0\pm 0} \int_{t_1}^{t_2} | (\partial u_i/\partial x)(x, t) | \, dt = \int_{t_1}^{t_2} \operatorname*{Limit}_{x=x_0\pm 0} | (\partial u_i/\partial x)(x, t) | \, dt,$$

*$x_0$ being the x-coördinate of any boundary.**

The proof of this theorem is the same as that of Theorem A except in the proof of the equation (f)

$$\operatorname*{Limit}_{x=x_0\pm 0} \int_{t_1}^{t_2} K_i V_i(x, t)(\partial V_i/\partial x)(x, t)\, dt = \int_{t_1}^{t_2} \operatorname*{Limit}_{x=x_0\pm 0} \{K_i V_i(x, t)(\partial V_i/\partial x)(x, t)\}\, dt.$$

In Theorem A this was a consequence of the boundedness of $V_i(x, t)$ and $\partial V_i(x, t)/\partial x$. Having removed this latter restriction, we need a different proof.

Each $V_i(x, t)$ is bounded, say less than $M_i$, hence

$$| K_i V_i(x, t)(\partial V_i/\partial x)(x, t) | \leq K_i M_i \, | (\partial V_i/\partial x)(x, t) |$$

and by the condition of our theorem

$$\operatorname*{Limit}_{x=x_0\pm 0} \int_{t_1}^{t_2} | (\partial V_i/\partial x)(x, t) | \, dt = \int_{t_1}^{t_2} \operatorname*{Limit}_{x=x_0\pm 0} | (\partial V_i/\partial x)(x, t) | \, dt$$

and so †

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* That is to say, the absolute continuity of $\int_0^t | (\partial u_i/\partial x)(x, t) | \, dt$—and hence of $\int_0^t (\partial u_i/\partial x)(x, t)\, dt$—is uniform in $x$. From this it follows that

$$\operatorname*{Limit}_{x=x_0\pm 0} \int_{t_1}^{t_2} (\partial u_i/\partial x)(x, t)\, dt = \int_{t_1}^{t_2} \operatorname*{Limit}_{x=x_0\pm 0} (\partial u_i/\partial x)(x, t)\, dt,$$

if $\operatorname*{limit}_{x=x_0\pm 0} (\partial u_i/\partial x)(x, t)$ exists.

† See Hobson, *Theory of Functions of a Real Variable*, Vol. 2 (1926), p. 290.

$$\operatorname*{Limit}_{x=x_0\pm 0}\int_{t_1}^{t_2} K_i V_i(x,t)\,(\partial V_i/\partial x)\,(x,t)\,dt = \int_{t_1}^{t_2} \operatorname*{Limit}_{x=x_0\pm 0}\{K_i V_i(x,t)\,(\partial V_i/\partial x)\,(x,t)\}\,dt.$$

This establishes the equation (f) and the remainder of the proof is unaltered.

*We will now show that a solution of the problem satisfying the conditions of the uniqueness theorem* B *is given by*

$$(2.6)\qquad u_1(x,t) = u_1 + \int_0^t \{(t-t')^{-\frac{1}{2}}\, e^{-a^2(x-m)^2/4(t-t')} + (t-t')^{-\frac{1}{2}} e^{-a^2(x+m)^2/4(t-t')}\}\,\psi_1(t')\,dt',$$

*for* $-m < x < m$,
*and*

$$(2.7)\qquad u_2(x,t) = u_2 + \int_0^t (t-t')^{-\frac{1}{2}} e^{-b^2(x-m)^2/4(t-t')}\psi_2(t')\,dt' + \int_0^t (t-t')^{-\frac{1}{2}} e^{-b^2(x-l)^2/4(t-t')}\psi_3(t')\,dt',$$

*for* $m < x < l$, $[u_2(x,t) = u_2(-x,t)$ for $-l < x < m]$, *where* $\psi_1(t)$, $\psi_2(t)$ *and* $\psi_3(t)$ *are summable functions satisfying the following integral equations, almost everywhere.*

$$(2.8)\qquad \int_0^t (t-t')^{-\frac{1}{2}}\psi_3(t')\,dt' = f_2(t) - \int_0^t (t-t')^{-\frac{1}{2}} e^{-\beta^2/(t-t')}\psi_2(t')\,dt'.$$

$$(2.9)\qquad \psi_2(t) = -c\psi_1(t) + c\alpha\pi^{-\frac{1}{2}}\int_0^t (t-t')^{-3/2} e^{-\alpha^2/(t-t')}\psi_1(t')\,dt' + \beta\pi^{-\frac{1}{2}}\int_0^t (t-t')^{-3/2} e^{-\beta^2/(t-t')}\psi_3(t')\,dt'.$$

$$(2.10)\qquad \int_0^t \{(t-t')^{-\frac{1}{2}} e^{-\alpha^2/(t-t')} + (t-t')^{-\frac{1}{2}}\}\,\psi_1(t')\,dt' = f_1 + \int_0^t (t-t')^{-\frac{1}{2}}\psi_2(t')\,dt' + \int_0^t (t-t')^{-\frac{1}{2}} e^{-\beta^2/(t-t')}\psi_3(t')\,dt'.$$

Here the integrals are taken to be Lebesgue integrals and we have used the following abbreviations:

(i) $f_2(t) = f(t) - u_2$
(ii) $\alpha = am$
(iii) $\beta = b(l-m)/2$
(iv) $c = K_1 a/K_2 b > 0$
(v) $f_1 = u_2 - u_1$

By actual differentiation we see that

$$(t - t')^{-\frac{1}{2}} e^{-a^2(x-x')/4(t-t')}$$

when considered as a function of $x$ and $t$ is a solution of the first equation (2. 1) for any $x' \neq x$ and for any $t' < t$. If we replace the $a^2$ in the exponent by $b^2$, we have a solution of the second equation (2. 1).*

The functions $u_1(x, t)$ and $u_2(x, t)$ as given by equations (2. 6) and (2. 7) are linear combinations of such solutions and except at the boundaries the integrals converge for any summable functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$. These expressions are easily seen to be continuous except at the boundaries. Likewise, if we form the first derivatives we get expressions which converge and are continuous except at the boundaries. The functions $u_1(x, t)$ and $u_2(x, t)$ thus satisfy the equation (2. 1) in the inner and outer regions respectively.

In each of the integrals the first factor is bounded except at the boundaries so that each integrand is summable and the limit of each integral, as $t$ approaches zero, is zero except at the boundaries. The initial conditions (2. 2) are thus satisfied.

As a consequence of equations (2. 8) to (2. 10) we can show that $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are equivalent to functions of the form

$$A_i/t^{\frac{1}{2}} + \phi_i(t)$$

where $\phi_i(t)$ is bounded and continuous.

Equation (2. 8) can be written

$$\text{(A)} \qquad \int_0^t (t - t')^{-\frac{1}{2}} \psi_3(t')\, dt' = g(t)$$

where

$$g(t) = f_2(t) - \int_0^t (t - t')^{-\frac{1}{2}} e^{-\beta^2/(t-t')} \psi_2(t')\, dt'.$$

In virtue of the conditions on $f(t)$ and the boundedness and continuity of $(t - t')^{-\frac{1}{2}} e^{-\beta^2/(t-t')}$ and the summability of $\psi_2(t)$, $g(t)$ is bounded and continuous. Its derivative

$$g'(t') = f'_1(t) - \int_0^t (\partial/\partial t) \{(t - t')^{-\frac{1}{2}} e^{-\beta^2/(t-t')}\} \psi_2(t')\, dt'$$

is likewise bounded and continuous.

Equation (A) is an Abel's equation of the usual type and has the solution †

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* For a full discussion of the properties of these and similar functions, see E. E. Levi, "Sull' equazione del calore," *Annali di Matematica* (3), Vol. 14 (1908), p. 187.

† Volterra, *loc. cit.*, p. 37.

$$\psi_3(t) = g(0)/\pi t^{\frac{1}{2}} + 1/\pi \int_0^t (t-t')^{-\frac{1}{2}} g'(t')\,dt'.$$

Since $g'(t)$ is bounded and continuous, the same is true of

$$\int_0^t (t-t')^{-\frac{1}{2}} g'(t')\,dt'$$

and $\psi_3(t')$ has the required form.

To show that $\psi_2(t)$ and $\psi_1(t)$ have this form, we eliminate $\psi_2(t)$ from equations (2.9) and (2.10). Thus from (2.9)

(B) $$\psi_2(t) = -c\psi_1(t) + h(t)$$

where

$$h(t) = c\alpha\pi^{-\frac{1}{2}} \int_0^t (t-t')^{-3/2}\, e^{-\alpha^2/(t-t')}\psi_1(t')\,dt' + \beta\pi^{-\frac{1}{2}} \int_0^t (t-t')^{-3/2}\, e^{-\beta^2/(t-t')}\psi_3(t')\,dt'$$

and since $\psi_1(t)$ and $\psi_3(t)$ are summable, it is bounded and continuous and has a bounded and continuous derivative. We have then

$$\int_0^t (t-t')^{-\frac{1}{2}}\psi_2(t')\,dt' = -c\int_0^t (t-t')^{-\frac{1}{2}}\psi_1(t')\,dt'$$

$$+\int_0^t (t-t')^{-\frac{1}{2}} h(t')\,dt' = -c\int_0^t (t-t')^{-\frac{1}{2}}\psi_1(t')\,dt'$$

$$+ c\int_0^t (t-t')^{-\frac{1}{2}}\, e^{-\alpha^2/(t-t')}\psi_1(t')\,dt + \int_0^t (t-t)^{-\frac{1}{2}}\, e^{-\beta^2/(t-t')}\psi_3(t')\,dt',$$

so that (2.10) gives

(C) $$\int_0^t (t-t')^{-\frac{1}{2}}\psi_1(t')\,dt' = h_1(t),$$

where

$$h_1(t) = f_1/(1+c) + 2/(1+c)\int_0^t (t-t')^{-\frac{1}{2}}\, e^{-\beta^2/(t-t')}\psi_3(t')\,dt' + (c-1)/(1+c)\int_0^t (t-t')^{-\frac{1}{2}}\, e^{-\alpha^2/(t-t')}\psi_1(t')\,dt',$$

and is bounded and continuous with a bounded and continuous derivative. So that, as above, equation (C) has the solution

$$\psi_1(t) = h_1(0)/\pi t^{\frac{1}{2}} + 1/\pi \int_0^t (t-t')^{-\frac{1}{2}} h'_1(t')\,dt,$$

and since $h_1'(t)$ is bounded and continuous, $\int_0^t (t-t')^{-\frac{1}{2}} h'_1(t')\,dt'$ is bounded and continuous and $\psi_1(t)$ has the required form.

From equation (B) is follows then that $\psi_3(t)$ also has this form.

Consider now a term of the form

$$\int_0^t (t-t')^{-\frac{1}{2}}\, e^{-a^2/(x-m)^2/4(t-t')}\psi_1(t')\,dt'.$$

In absolute value this is less than

$$\int_0^t (t-t')^{-\frac{1}{2}} \mid \psi_1(t') \mid dt'$$
$$\leq \int_0^t \pi^{-1}(t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}} \mid h_1(0) \mid dt' + \int_0^t (t-t')^{-\frac{1}{2}}\,\phi_1(t')\,dt'$$

and so is bounded. Hence $u_1(x,t)$ and $u_2(x,t)$, being the sum of such terms and a finite constant, are bounded.

We also have

$$\underset{x=l-0}{\text{Limit}} \int_0^t (t-t')^{-\frac{1}{2}}\, e^{-b^2(x-l)^2/4(t-t')}\psi_3(t')\,dt' = \int_0^t (t-t')^{-\frac{1}{2}}\psi_3(t')\,dt'$$

since the integrand is less than a summable function, independent of $x$. We have then

$$\underset{x=l-0}{\text{Limit}}\ u_2(x,t) = u_2$$
$$+ \int_0^t (t-t')^{-\frac{1}{2}}\, e^{-\beta^2/(t-t')}\psi_2(t')\,dt' + \int_0^t (t-t')^{-\frac{1}{2}}\,\psi_3(t')\,dt' = f(t)$$

in virtue of equation (2. 8). Thus the boundary condition (2. 3) is satisfied.

Now in the same way we can show

$$\underset{x=m+0}{\text{Limit}}\ u_2(x,t) = u_2$$
$$+ \int_0^t (t-t')^{-\frac{1}{2}}\,\psi_2(t')\,dt' + \int_0^t (t-t')^{-\frac{1}{2}}\, e^{-\beta^2/(t-t')}\psi_3(t')\,dt'$$

and

$$\underset{x=m-0}{\text{Limit}}\ u_1(x,t) = u_1 + \int_0^t \{(t-t')^{-\frac{1}{2}}\, e^{-a^2/(t-t')} + (t-t')^{-\frac{1}{2}}\}\,\psi_1(t')\,dt',$$

so that from equation (2. 10), recalling (v), we have

$$\underset{x=m-0}{\text{Limit}}\ u_1(x,t) = \underset{x=m+0}{\text{Limit}}\ u_2(x,t)$$

and the boundary condition (2. 4) is satisfied.

To show that the boundary condition (2. 5) is satisfied we need to know the value of

$$\underset{x=m+0}{\text{Limit}}\ (\partial u_2/\partial x)(x,t).$$

By direct differentiation we have

$$(\partial u_1/\partial x)(x,t) = -a^2/2 \int_0^t \{(t-t')^{-3/2}(x+m)\, e^{-a^2(x+m)^2/4(t-t')}$$
$$+ (t-t')^{-3/2}(x-m)\ e^{-a^2(x-m)^2/4(t-t')}\}\,\psi_1(t')\,dt',$$

and

$$(\partial u_2/\partial x)(x,t) = -b^2/2\int_0^t (t-t')^{-3/2}(x-m)e^{-b^2(x-m)^2/4(t-t')}\psi_2(t')\,dt'$$
$$-b^2/2\int_0^t (t-t')^{-3/2}(x-l)e^{-b^2(x-l)^2/4(t-t')}\psi_3(t')\,dt'.$$

The integrals involved all converge, except at the boundaries. To evaluate $\underset{x=m+0}{\text{Limit}}\ (\partial u_2/\partial x)(x,t)$, we need to know the value of

$$\underset{x=m+0}{\text{Limit}}\ b/2\pi^{1/2}\int_0^t (t-t')^{-3/2}(x-m)e^{-b^2(x-m)^2/4(t-t')}\phi(t')\,dt',$$

for $\phi(t)$ summable. Hobson * considers a more general problem. Restating his results in terms applicable here, we have the Theorem:

Let $F(t',t,x)$ be defined for $t', t>0$ and $m<x<l$. Let $\mu$ denote a positive number and let $F(t',t,x)$ satisfy the following conditions;

(1) For each pair of values $t$ and $x$, and for all values of $t'>0$, such that $|t-t'|\geq\mu$, the function $F(t',t,x)$ is equivalent to a function that does not exceed in absolute value a positive number $K_\mu$, independent of the values of $t$ and $x$.

(2) If $\alpha$ and $\beta$ are two numbers such that

$$0\leq\alpha\leq\beta\leq t, \int_\alpha^\beta F(t',t,x)\,dt'$$

exists as a Lebesgue integral, for all values of $x$, $(m<x<l)$, and for all those values of $t$ such that $t-\mu>\alpha,\beta$; and as $x$ approaches $m+0$, the integral converges to zero, uniformly for all such values of $t$.

Let $F(t',t,x)$ be a function of $t-t'$, say $F(t-t',x)$ and satisfy the conditions;

(a) $\underset{x=m-0}{\text{Limit}}\int_0^\mu F(z,x)\,dz = 1$

(b) $\int_0^\mu |F(z,x)|\,dz < A$

where $A$ is independent of $\mu$ and $x$, for all sufficiently small values of $\mu>0$.

(I) $\underset{x=m+0}{\text{Limit}}\ F(t-t',x)=0$, when $t'\neq t$.

(II) $zF(z,x)$ has a total variation in the interval $(0,\mu)$ less than a fixed number independent of $x$.

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* Hobson, *Theory of Functions of a Real Variable*, Vol. 2 (1926), pp. 443, 447 and 452.

The conclusion of the theorem is, that if these six conditions are satisfied by $F(t-t', x)$,

$$\underset{x=m+0}{\text{Limit}} \int_0^t F(t-t', x)\phi(t')\,dt' = \phi(t)$$

whenever

$$\int_0^{t'} \{\phi(t+t'') - \phi(t)\}\,dt''$$

has a differential coefficient, with respect to $t'$, equal to zero for $t'=0$. This is the case for almost all values of $t$, since $\phi(t)$ is summable.

We can easily show that the function

$$b/2\pi^{\frac{1}{2}}(t-t')^{-3/2}(x-m)e^{-b^2(x-m)^2/4(t-t')}$$

satisfies these conditions and so

$$\underset{x=m+0}{\text{Limit}}\ b/2\pi^{\frac{1}{2}} \int_0^t (t-t')^{-3/2}(x-m)e^{-b^2(x-m)^2/4(t-t')}\phi(t')\,dt' = \phi(t)$$

almost everywhere, if $\phi(t)$ is summable.

Similarly we can show

$$\underset{x=m-0}{\text{Limit}}\ (-a/2\pi^{\frac{1}{2}}) \int_0^t (t-t')^{-3/2}(x-m)e^{-a^2(x-m)^2/4(t-t')}\phi(t')\,dt' = \phi(t)$$

almost everywhere, if $\phi(t)$ is summable.

Applying these results to equation (2. 12) we have

$$\underset{x=m-0}{\text{Limit}}\ \partial u_1(x,t)/\partial x = a\pi^{\frac{1}{2}}\psi_1(t) - \int_0^t a^2 m(t-t')^{-3/2}\, e^{-a^2/(t-t')}\psi_1(t')\,dt'$$

and similarly from (2. 13) we have

$$\underset{x=m+0}{\text{Limit}}\ \partial u_2(x,t)/\partial x = -b\pi^{\frac{1}{2}}\psi_2(t) - \int_0^t [\tfrac{1}{2}b^2(m-l)(t-t')^{-3/2}]\, e^{-\beta^2/(t-t')}\psi_2(t')\,dt'.$$

In virtue of equation (2. 9) we have then

$$\underset{x=m-0}{\text{Limit}}\ K_1(\partial u_1/\partial x)(x,t) = \underset{x=m+0}{\text{Limit}}\ K_2(\partial u_2/\partial x)(x,t)$$

and the boundary condition (2. 5) is satisfied.

We need now to prove that

$$\underset{x=x_0\pm 0}{\text{Limit}} \int_{t_1}^{t_2} |\,\partial u_i/\partial x)(x,t)\,|\,dt = \int_{t_1}^{t_2} \underset{x=x_0\pm 0}{\text{Limit}}\ |\,\partial u_i/\partial x(x,t)\,|\,dt,$$

$x_0$ being the $x$ — coördinate of any boundary.

As a typical case we consider

$$\underset{x=-m+0}{\text{Limit}} \int_{t_1}^{t_2} | (\partial u_1/\partial x)(x,t) | \, dt$$

$$= \underset{x=-m+0}{\text{Limit}} \int_{t_1}^{t_2} | -a^2/2 \int_0^t (t-t')^{-3/2}(x-m) e^{-a^2(x-m)^2/4(t-t')} \psi_1(t') \, dt'$$

$$- a^2/2 \int_0^t (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} \psi_1(t') \, dt' | \, dt.$$

The term

$$G(x,t) = -a^2/2 \int_0^t (t-t')^{-3/2}(x-m) e^{-a^2(x-m)^2/4(t-t')} \psi_1(t') \, dt$$

is bounded and continuous in $x$ as $x$ approaches $-m$, so that

$$(\alpha) \qquad \underset{x=-m+0}{\text{Limit}} \int_{t_1}^{t_2} | G(x,t) | \, dt = \int_{t_1}^{t_2} \underset{x=-m+0}{\text{Limit}} | G(x,t) | \, dt.$$

Consider the term

$$H(x,t) = -a^2/2 \int_0^t (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} \psi_1(t') \, dt'.$$

We define

$$H(x,t) = a^2/2 \int_0^t (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} | \psi_1(t') | \, dt'$$

and by the theorem of Hobson quoted above

$$\underset{x=-m+0}{\text{Limit}} \bar{H}(x,t) = (\pi^{1/2}/a) | \psi_1(t) |$$

nearly everywhere, since $| \psi_1(t) |$ is summable.

We have $\bar{H}(x,t) \geq | H(x,t) |$ for all $x$ and $t$.

Consider now

$$\underset{x=-m+0}{\text{Limit}} \int_0^T \bar{H}(x,t) \, dt$$

$$= \underset{x=-m+0}{\text{Limit}} \int_0^T dt \, (a^2/2) \int_0^t (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} | \psi_1(t') | \, dt'.$$

By a change of order of integration we have

$$\int_0^T dt \, (a^2/2) \int_0^t (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} | \psi_1(t') | \, dt'$$

$$= \int_0^T dt' | \psi_1(t') | \, a^2/2 \int_{t'}^T (t-t')^{-3/2}(x+m) e^{-a^2(x+m)^2/4(t-t')} \, dt$$

$$= \int_0^T g(t', T, x) | \psi_1(t') | \, dt'$$

where *

$$g(t', T, x) = a^2/2 \int_{t'}^{T} (t - t')^{-3/2}(x + m) e^{-a^2(x+m)^2/4(t-t')} dt, \qquad x > -m$$

$$\leq a^2/2 \int_{t'}^{\infty} (t - t')^{-3/2}(x + m)\, e^{-a^2(x+m)^2/4(t-t')} dt$$

Putting $\xi = a(x + m)/2(t - t')^{1/2}$ we have

$$g(t', T, x) \leq 2a \int_0^{\infty} e^{-\xi^2} d\xi = a\pi^{1/2} \text{ independent of } t', T, x.$$

We have then

$$\underset{x=-m+0}{\text{Limit}} \int_0^T g(t', T, x) \mid \psi_1(t') \mid dt = \int_0^T \underset{x=-m+0}{\text{Limit}}\ g(t', T, x) \mid \psi_1(t') \mid dt'$$

$$= a\pi^{1/2} \int_0^T \mid \psi_1(t') \mid dt = \int_0^T \underset{x=-m+0}{\text{Limit}}\ \bar{H}(x, t')\, dt'$$

That is

$$\underset{x=-m+0}{\text{Limit}} \int_0^T \bar{H}(x, t)\, dt = \int_0^T \underset{x=-m+0}{\text{Limit}}\ \bar{H}(x, t)\, dt$$

And, since $\bar{H}(x, t) \geq \mid H(x, t) \mid \geq 0$, and since Limit $H(x, t)$ exists nearly everywhere, we have

$$\underset{x=-m+0}{\text{Limit}} \int_0^T \mid H(x, t) \mid dt = \int_0^T \underset{x=-m+0}{\text{Limit}}\ \mid H(x, t) \mid dt$$

Hence recalling $(\alpha)$ we have

$$\underset{x=-m+0}{\text{Limit}} \int_0^T \mid (\partial u_1/\partial x)(x, t) \mid dt = \int_0^T \underset{x=-m+0}{\text{Limit}} \mid (\partial u/\partial x)(x, t) \mid dt$$

as was to be shown.†

*Thus the solution given by $u_1(x, t)$ and $u_2(x, t)$ in the inner and outer regions respectively satisfies the conditions of the Uniqueness Theorem B.*

It remains to be shown that $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ can be found to satisfy equations (2. 15) to (2. 17). This will be done in the next two sections.

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* For $T > t'$ $\underset{x=-m+0}{\text{Limit}}\ g(t', T, x) = a\pi^{1/2}$.

† We have used the limits 0 and $T$, the equation holds for any limits $t_1$ and $t_2$.

## PART III. Case of Infinite Side Regions.

A case of considerable interest is the one where the outer regions are infinite.* In this section we give a method of solution of this case.

Here we have no outer boundary and replace the condition (2. 3) by the condition that $\underset{x=\infty}{\text{Limit}}\,[\partial u_2(x,t)/\partial x] \equiv 0$. The solution for the outer region differs from that above by the absence of the term involving $\psi_3(t)$. The result is that we have two integral equations for the two functions $\psi_1(t)$ and $\psi_2(t)$. These are

$$\psi_2(t) = -c\psi_1(t) + c\alpha\pi^{-\frac{1}{2}} \int_0^t (t-t')^{-3/2} e^{-\alpha^2/(t-t')} \psi_1(t')\,dt' \tag{3.1}$$

and

$$\int_0^t \{(t-t')^{-\frac{1}{2}} e^{-\alpha^2/(t-t')} + (t-t')^{-\frac{1}{2}}\}\,\psi_1(t')\,dt' = f_1 + \int_0^t (t-t')^{-\frac{1}{2}} \psi_2(t')\,dt' \tag{3.2}$$

If we substitute the value of $\psi_2(t)$ given by the right-hand member of (3. 1) into (3. 2), change the order of integration and carry out the integration in the inner integrals, we have

$$\int_0^t \{(t-t')^{-\frac{1}{2}}[1 + (\overline{1-c}/\overline{1+c})e^{-\alpha^2/(t-t')}]\}\,\psi_1(t')\,dt' = f_1/(1+c) \tag{3.3}$$

This is an equation of the sort dealt with in Part I with

$$K(t,t') = (1-c/1+c)\,e^{-\alpha^2/(t-t')}$$

where $c$ is a positive constant. Since $K(t,t')$ vanishes exponentially for $t = t'$ and is bounded elsewhere, the conditions of Part I are satisfied and we can find a summable solution of (3. 3) by the method given in that part. This solution, $\psi_1(t)$, substituted in (3. 1) gives a summable function $\psi_2(t)$. These substituted in (2. 9) and the modified (2. 7) give functions $u_1(x,t)$ and $u_2(x,t)$ that solve the problem.

## PART IV. Solution of Equations for Finite Side Regions.

In this section we consider the more general case of finite outer regions. As in Part III, we do not solve for $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ directly, but, assuming the existence of summable solutions, $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, for

* This problem is treated by A. Sommerfeld, "Zur analytischen Theorie der Wärmeleitung," *Mathematische Annalen*, Vol. 45 (1894), p. 270, by constructing the Green's function by the method of images.

$\int_0^{t''}\psi_1(t)\,dt$, $\int_0^{t''}\psi_2(t)\,dt$ and $\int_0^{t''}\psi_3(t)\,dt$. In order to do this we must make changes in the equations (2. 15), (2. 16) and (2. 17) so that they will involve the integrals.

We multiply equation (2. 15) by $(t''-t)^{-\frac{1}{2}}$ and integrate from $t=0$ to $t=t''$, thus

$$(4.1)\quad \int_0^{t''}(t''-t)^{-\frac{1}{2}}dt\int_0^{t}(t-t')^{-\frac{1}{2}}\psi_3(t')\,dt' = \int_0^{t''}(t''-t)^{-\frac{1}{2}}f_2(t)\,dt - \int_0^{t''}(t''-t)^{-\frac{1}{2}}dt\int_0^{t}(t-t')^{-\frac{1}{2}}e^{-\beta^2/(t-t')}\psi_2(t')\,dt'$$

The first term by a change of order of integration gives

$$\pi\int_0^{t''}\psi_3(t')\,dt'.$$

Integrating the inner integral of the last term by parts gives

$$\left[(t-t')^{-\frac{1}{2}}e^{-\beta^2/(t-t')}\int_0^{t'}\psi_2(t'')\,dt''\right]_{t'=0}^{t'=t} - \int_0^{t}\left\{\int_0^{t'}\psi_2(t'')\,dt''\right\}\left[(t-t'-2\beta^2)/2\,(t-t')^{5/2}\right]e^{-\beta^2/(t-t')}\,dt'$$

The term outside the sign of integration is zero. Putting this in and changing the order of integration gives, in place of (4. 1)

$$(4.2)\quad \pi\int_0^{t''}\psi_3(t')\,dt' = \int_0^{t''}(t''-t')^{-\frac{1}{2}}f_2(t')\,dt' + \int_0^{t''}\left\{\int_0^{t'}\psi_2(t)\,dt\right\}dt'\int_{t'}^{t''}\left[(t-t'-2\beta^2)/2(t-t')^{5/2}\right]e^{-\beta^2/(t-t')}\,dt'$$
$$= \int_0^{t''}(t''-t')^{-\frac{1}{2}}f_2(t')\,dt' - \int_0^{t''}\left\{\int_0^{t'}\psi_2(t)\,dt\right\}\left[\beta/(t''-t')^{3/2}\right]\,e^{-\beta^2/(t''-t')}\,dt'.$$

We integrate the equation (2. 16) from $t=0$ to $t=t''$ and in the last two terms integrate by parts and then change the order of integration, this gives

$$(4.3)\quad \int_0^{t''}\psi_2(t')\,dt' = -c\int_0^{t''}\psi_1(t')\,dt' - c\alpha\pi^{-\frac{1}{2}}\int_0^{t''}\left\{\int_0^{t'}\psi_3(t)\,dt\right\}dt'\int_0^{t''}(\partial/\partial t')\left[(t-t')^{-3/2}\,e^{-\alpha^2/(t''-t')}\right]dt - \beta\pi^{-\frac{1}{2}}\int_0^{t''}\left\{\int_0^{t'}\psi_1(t)\,dt\right\}dt'\int_{t'}^{t''}(\partial/\partial t')\left[(t-t')^{-3/2}\,e^{-\beta^2/(t''-t')}\right]dt$$

$$= -c\int_0^{t''}\psi_1(t')\,dt'$$

$$+ c\alpha\pi^{-1/2}\left\{\int_0^{t''}\int_0^{t'}\psi_1(t)\,dt\right\}(t''-t)^{-3/2}\,e^{-\alpha^2/(t''-t')}\,dt'$$

$$+ \beta\pi^{-1/2}\int_0^{t''}\left\{\int_0^{t''}\psi_3(t)\,dt\right\}(t''-t')^{-3/2}e^{-\beta^2/(t''-t')}\,dt.$$

Multiplying equation (2.17) by $(t''-t)^{-1/2}$ and integrating from $t=0$ to $t=t''$, we obtain, in a manner similar to that used in deriving equation (4.2),

$$\alpha\int_0^{t''}\left\{\int_0^{t'}\psi_1(t)\,dt\right\}(t''-t')^{-3/2}\,e^{-\alpha^2/(t''-t')}\,dt' \tag{4.4}$$

$$+\pi\int_0^{t''}\psi_1(t')\,dt' = 2f_1(t'')^{-1/2} + \pi\int_0^{t''}\psi_2(t')\,dt'$$

$$+\beta\int_0^{t''}\left\{\int_0^{t'}\psi_3(t)\,dt\right\}(t''-t')^{-3/2}\,e^{-\beta^2/(t''-t')}\,dt'.$$

By putting

$$F(t'') = \int_0^{t''}(t''-t')^{-1/2}f_2(t')\,dt'$$

$$V_i(t'') = \int_0^{t''}\psi_i(t)\,dt \qquad (i=1,2,3)$$

$$H_\rho(t'',t') = [\rho/(t''-t')^{3/2}]\,e^{-\rho^2/(t''-t')} \qquad (\rho=\alpha,\beta)$$

we can write (4.2), (4.3) and (4.4) as

$$\pi V_3(t'') = F(t'') - \int_0^{t''}V_2(t')H_\beta(t'',t')\,dt' \tag{4.5}$$

$$V_2(t'') = -c\,V_1(t'') + h\int_0^{t''}V_1(t')H_\alpha(t'',t')\,dt' \tag{4.6}$$

$$+\pi^{-1/2}\int_0^{t''}V_3(t')H_\beta(t'',t')\,dt'$$

and

$$\int_0^{t''}V_1(t')H_\alpha(t'',t')\,dt' + \pi V_1(t'') \tag{4.7}$$

$$= 2f_1(t'')^{-1/2} + \pi V_2(t'') + \int_0^{t''}V_3(t')H_\beta(t'',t')\,dt'.$$

Equation (4.6) multiplied by $\pi$ and added to equation (4.7) gives

$$V_1(t'') = 1/\pi(1+c)\{2f_1(t'')^{1/2} \tag{4.8}$$

$$+(c\pi^{1/2}-1)\int_0^{t''}V_1(t')H_\alpha(t'',t')\,dt'$$

$$+(1+\pi^{1/2})\int_0^{t''}V_3(t')H_\beta(t'',t')\,dt'\}.$$

Equation (4. 6) multiplied by $\pi$ and subtracted from $c$ times equation (4. 7) gives

$$(4.9)\qquad V_2(t'') = 1/\pi(1+c)\{-2cf_1(t'')^{\frac{1}{2}} + c(1+\pi^{\frac{1}{2}})\int_0^{t''} V_1(t')H_\alpha(t'',t')\,dt' - (c-\pi^{\frac{1}{2}})\int_0^{t''} V_3(t')H_\beta(t'',t')\,dt'\}$$

We have in equations (4. 5), (4. 8) and (4. 9) a system of Volterra integral equations of the second kind with bounded kernels. Such a system is solvable by a process of successive approximations that *necessarily* converges.* It may be pointed out that, since the kernels involved vanish exponentially, the process of approximation converges rapidly.

The terms on the right-hand sides of equations (4. 5), (4. 8) and (4. 9) are of the types shown to be absolutely continuous in Part I and so the functions $V_1(t)$, $V_2(t)$ and $V_3(t)$ are absolutely continuous. They thus possess derivatives nearly everywhere and are the integrals of these derivatives. We wish to show that the functions $\psi_1(t) = V_1'(t)$, $\psi_2(t) = V_2'(t)$ and $\psi_3(t) = V_3'(t)$ satisfy the equations (2. 15), (2. 16) and (2. 17).

We do this by a device similar to that employed in Part I. That is we substitute $V_2'(t)$ for $\psi_2(t)$ and $V_3'(t)$ for $\psi_3(t)$ in the right- and left-hand members of equation (2. 15) and call the difference between the two members $D_1(t)$. Then multiplying by $(t''-t)^{-\frac{1}{2}}$ and integrating from $t=0$ to $t=t''$, changing order of integration and integrating by parts as was done to obtain equations (4. 1) and (4. 2) we have

$$\int_0^{t''}(t''-t)^{-\frac{1}{2}}D(t)\,dt = \pi\int_0^{t''}V_3'(t)\,dt - F(t'') + \beta\int_0^{t''}\left\{\int_0^{t''}V_2'(t)\,dt\right\}H_\beta(t'',t')\,dt'$$

which is zero since $\int_0^{t'}V_2'(t)\,dt = V_2(t')$ and $\int_0^{t'}V_3'(t)\,dt = V_3(t')$ and satisfy equation (4. 6). Hence $D_1(t)$ is zero almost everywhere and equation (2. 15) is satisfied.

Similarly equations (2. 16) and (2. 17), by the methods used to obtain equations (4. 3) and (4. 5), are shown to be satisfied almost everywhere in virtue of the equations (4. 9) and (4. 10).

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* See Volterra, *Leçons sur les Équations Intégrales*, p. 71.

## PART V. Extension to General Cases.

The method of Part IV will apply to any number of regions and to conditions which are not symmetric. The only merit of the symmetric conditions is that they reduce the number of conditions by half.

The initial temperatures were taken to be constant in order to make the form of equations (2.7) and (2.9) as simple as possible. If the initial temperatures are not constant, the constant $u_1$ in equation (2.9) is replaced by the expression

$$a/2\pi^{\frac{1}{2}} \int_{-m}^{+m} u_1(x')\, t^{-\frac{1}{2}}\, e^{-a^2(x-x')^2/4t} dx'$$

where $u_1(x)$ is the initial temperature. This expression satisfies equation (2.1) and approaches $u_1(x)$, as $t$ approaches zero, nearly everywhere if $u_1(x)$ is summable.* For varying initial temperature in the outer regions the constant $u_2$ is replaced by an analogous expression involving $u_2(x)$, the initial temperature in the outer region. We can show that this solution satisfies the conditions of the Uniqueness Theorem B if $u_1(x)$ and $u_2(x)$ are bounded. The introduction of these expressions necessitates the use of equations more complicated than (2.8) to (2.10), but makes no essential difference in the method.

In closing I should like to acknowledge my indebtedness to Dr. G. C. Evans, both for his proposal of this problem and for his timely criticisms and suggestions.

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* Hobson, *loc. cit.*, Vol. 2, p. 402.